Algorithmic Game Theory	**Problem Set 1**
CS 684 Spring 2008	**Due in class Thursday, February 21, 2008**

We will maintain a FAQ for the problem set on the course Web page. You may use any fact we proved in class without proving the proof or reference. However, you may not use other published papers, or the Web to find your answer.

You are expected to attempt all problems, but its OK if you cannot solve them all. A full solution for each problem includes proving that your answer is correct. If you cannot solve a problem, write down how far you got, and why are you stuck. You may solve the problem with a partner. Please hand in a shared problem set with both your names on the solutions.

We are happy to help finding partners, let us (Thanh or Eva) know if you don't have a partner, and would like one.

(1) In class we considered a resource allocation game where n players are allocating a shared bandwidth of 1. Each player i chooses an amount $x_i \geq 0$ and the utility of player i is $U_i(x_i, x_{-i}) = x_i(1 - \sum_j x_j)$. Note that this utility is a bit different from the one in class, if $\sum_j x_j > 1$ we will have all players with negative utility.

(a) Does this game have a Nash equilibrium? If it does, give a Nash equilibrium.

(b) Is the Nash equilibrium unique? Prove your answer?

(c) Recall that a game is a potential game if there is a function $\Phi(x)$ such that for any state x of the game, any player i, and any alternate strategy x'_i for this player $\Phi(x'_i, x_{-i}) - \Phi(x_i, x_{-i}) = U_i(x'_i, x_{-i}) - U_i(x_i, x_{-i})$. Is the bandwidth sharing game a potential game?

(2) Consider a 2-player game where both players have n strategies. Such a game is defined by $2n^2$ numbers, the payoffs of the game for the two players on the n^2 possible strategy pairs the players can choose. Now assume this is a random game, where each of these $2n^2$ numbers is independently chosen uniformly random in the range $[0, 1]$

Show that the probability that this game has a pure Nash equilibrium is at least 60% as n gets high enough. You may use that $\lim(1 - 1/n)^n = 1/e$.

(3) Here we consider a simpler version of the grouping game from class defined as follows. Players are nodes of a graph, and edges represent social relations. We assumed that edge pair of nodes i and j has one associated with the pair $u_{ij} \geq 0$, where $u_{ij} \geq 0$ is the benefit players i and j both incur if nodes i and j are together. (In class we also had a benefit for not being together with people, here we forgot about the second part). In the game each player can choose between affiliating with one of two "parties" A or B. The result is a partition of the nodes into two sets. The utility of a player $i \in A$ is $U_i(A, B) = \sum_{j \in A} u_{ij}$. If player i is in B its utility is $U_i(A, B) = \sum_{k \in B} u_{ik}$. We assumed that the values are symmetric, namely that $u_{ij} = u_{ji}$ for all i and j. We used social welfare $\sum_i U_i(A, B)$ as the quality measure of a solution. We showed that this game is a (1) potential game, hence (2) it has a pure Nash equilibrium, (3) the socially optimal solution is a Nash, and (4) the price of anarchy is at least 1/2. This question explores to what extend did the results depend on some of the simplifying modeling assumptions we used. Consider the changes in the game proposed by each part separately.

(a) Which of the claims (1-4) remain true if there are three parties A, B, and C?

(b) Which of the claims (1-4) remain true if we modify the game to add a component of utility that depends on whether the player joins A or B. You can do this for example by adding values $a_i, b_i \geq 0$ that are user i's valuations of joining the party A and B respectively.

(c) Which of the claims (1-4) remain true if we no longer assume symmetry, that is do not assume that $u_{ij} = u_{ji}$ for all i and j.

(d) We showed in class that the original game is a potential game. It turns out that all potential games are congestion games with the right definition of congestion. Define a congestion game that is equilivalent to this grouping game.

(4) Consider a load balancing game with n jobs and m machines. Job j ha size p_j, and we are looking to assign each job to one of the available machines. In this problem, assume that all jobs can be assigned to any of the m machines. As usual let $r_i(x)$ be the response time of machine i if the total load (sum of sizes of assigned jobs) is x. Assume $r_i(x) = x$ for all machines.

(a) Is this a potential game?

(b) Consider the objective function of minimizing the maximum response time. Show that the maximum response time in any Nash equilibrium is at most twice the maximum response time in any other assignment.

(c) Is a bound better than 2 possible here? For example, is it true that all Nash equilibria have minimum possible response time?

(5) Consider the cost-sharing game we discussed in class, where each player i selects a subgraph of edges F_i connecting its terminals, and if $c_e \geq 0$ is the cost of edge e, and k_e is the number of players that selected edge e in a Nash solution, than the cost of player i is $Cost(i) = \sum_{e \in F_i} c_e/k_e$. We have seen that this is a potential game, and has an H_k bound on the price of stability if the game has k players. Show that the price of anarchy of this class of games is k.

(6) Consider the non-atomic multicommodity flow problem we have been discussing in class (defined on February 7th). We defined Nash equilibria as a flow f such that for all pairs of paths P and Q connecting the same pair of terminals, if $f_P > 0$ then $\ell_P(f) \leq \ell_Q(f)$.

A maybe more intuitive definition would be as follows. A flow f is a Nash equilibrium, if the following holds. For any pairs of paths P and Q connecting the same pair of terminals, such that $f_P > 0$ and any $0 < \delta \leq f_P$ if we define an alternate flow $\hat{f}$ by setting

$$\hat{f}_R = \begin{cases} f_P - \delta & \text{if } R = P \\ f_Q + \delta & \text{if } R = Q \\ f_R & \text{otherwise} \end{cases}$$

then $\ell_P(f) \leq \ell_Q(\hat{f})$.

This definition considers the flow f and a very small δ amount of flow on a path P, and wonders if this small amount of flow is happier on path P or should it switch to another path Q. Here we model the flow being non-atomic by allowing arbitrary small amounts of flow to switch, but we do not allow "zero" amount to switch, as it is less clear what that means.

Show that the two definitions are the same.