Algorithmic Game Theory	**Problem Set 2**
CS 684 Fall 2005	**Due in class Wednesday, October 19, 2005**

We will maintain a FAQ for the problem set on the course Web page. You may use any fact we proved in class without proving the proof or reference. However, you may not use published papers.

You are expected to attempt all problems. If you cannot solve a problem, write down how far you got, and why are you stuck.

Cooperation in developing answers is encouraged. However, each student must write down all answers separately.

(1) (Question (6) from the last problem set. Hint: recall the relation of the potential function Φ and the total delay for this case.)

Consider the one commodity special case of the nonatomic selfish routing game discussed in class where all traffic goes from a common source s to a common destination t. Assume for this problem that the delay on each edge is a nonnegative, linear and monotone increasing function of the load.

For this problem we define a flow f^* to be optimal if the longest paths that carries flow is as short as possible, and we define a flow to be *fair* if all flow is carried on equal length paths. (This definition assumes that users realize the existence of a better path only by seeing other users who use that path, and the length of path not carrying flow is not relevant for the definition.) We know from class (essentially by definition) that the Nash flow is fair. From the Braess paradox example, we also see that there can be a fair flow that is better than the flow at Nash equilibrium.

(a) Prove that the Nash flow is at most a factor of 4/3 worse than the optimal for the objective of minimizing the longest path carrying flow.

(b) For this part consider a flow f^* that minimizes average delay, that is, minimizes $\sum_P f_P d_P(f)$. We know that this optimal flow may not be fair. We measure the unfairness of this flow by the ratio of the lengths of the longest and shortest (s, t) paths that carries flow. Prove that the unfairness of the flow f^* is at most 2.

(2) In the Vetta facility location game we showed that if the facilities cost 0, and each player is allowed to locate one facility, than for any Nash equilibrium we get that $value(OPT) \leq 2 \cdot value(Nash)$, where OPT denotes the solution of maximum value.

In this problem, we consider a variant of this game where facilities cost money. In this variant, each possibly facility j has a cost c_j, and locating a facility at j the player has to pay cost c_j. (We define prices p_i as before, and in defining the value derived by a player who located at facility j, we use $\sum(p_i - \lambda_{ij}) - c_j$, where the sum is over the locations that will be served by facility j, so we simply subtract c_j from the value used in the game in class).

(a) Is the inequality $value(OPT) \leq 2 \cdot value(Nash)$ also true for the variant of this game that facilities cost money? Prove or give an example where it is not true.

(b) Let $fac(OPT)$ denote the total facility cost of optimum, that is the sum $\sum c_j$ over the facilities j opened by the optimal solution. This can be viewed as the investment cost needed for the optimal solution. Show that in any Nash equilibrium of this game satisfies $value(OPT) \leq 2 \cdot value(Nash) + fac(Nash)$. One can read this inequality to say that the price of anarchy can be bad (much worse than a factor of 2) only when the investment cost needed for the optimal solution is high.

- (Open problem.) Is the inequality $value(OPT) \leq 2 \cdot value(Nash) + fac(Opt)$ also true?

(3) On September 28 and 30 we considered a utility game. Recall the 3 rules for a utility game are using the notation from lecture.

(a) The resulting utility $value(S)$ for any set S of possible actions satisfies the decreasing marginal utility property: For any sets $A \subset B$ and action $s \notin B$, we have $value(B+s) - value(B) \leq value(A+s) - value(A)$.

(b) For a set of actions S that consists of one action s_k for each player k, if $\alpha_k(S)$ denotes the value derived by player k from this set of actions, then we have $value(S) \geq \sum_k \alpha_k(S)$.

(c) For a set of actions S that consists of one action s_k for each player k, we have $\alpha_k(S) \geq value(S) - value(S - s_k)$.

Recall that the last property was called (3') in lecture, where the stronger property (3) requiring equation defined basic utility games.

Now consider a variant of the selfish routing game. Assume that we have a graph $G = (V, E)$, and a delay function $\ell_e(x)$ on each edge $e \in E$. Assume for this problem that $\ell_e(x)$ is a monotone increasing and convex function of x for each edge e. Now we have k players $i = 1, 2, \ldots, k$ as usual, with a source s_i and a destination t_i. Assume that each player i needs to select a path P_i from its source s_i to its destination t_i, and needs to route 1 unit of flow along this path. The traditional player objective function in this game is to minimize delay. However, now we assume, each player has a maximum delay d_i that he is willing to tolerate. In a Nash equilibrium, we allow a player not to have any path at all, if all paths would have delay at least d_i.

(a) Show that this game always has a pure (deterministic) Nash equilibrium.

(b) The traditional way we evaluated such routing games is with the sum of all delays as cost. However, in this version, this cost can be low simply by not routing the players. Instead, we can consider value: namely, if a player selected a path P_i, and x_e denotes the number of players on edge e, then $\sum_{e \in P_i} \ell_e(x_e)$ is the delay experienced by this user, and let $\alpha_i(x) = d_i - \sum_{einP_i} \ell_e(x_e)$ be the resulting value for the user. By definition, all players routed have nonnegative value, and we define $\alpha_i(x) = 0$ for a player i that has no path in solution x. The total value of a solution x of this game is $\sum_i \alpha_i(x)$. Show that this is a utility game.

(c) What would be a strong enough monotonicity assumption, that allows us to use Vetta's argument to show that the total value of any Nash equilibrium in this game is at least 1/2 of the maximum possible value.

- (Open problem). Is there a way to derive an analog for this game (or the nonatomic version) of the bicriteria bound that Nash cost no more than the optimum routing twice as much flow?

(4) In the Johari -Tsitsiklis allocation game, covered in class on Friday, October 7th, considers a graph $G = (V, E)$ where each player i has a path P_i, and he wants to reserve bandwidth along the edges of P_i. Each edge has a capacity b_e, and each user has a utility function $U_i(x)$, which we assumed is strictly increasing, strictly concave and differentiable.

In the Johari -Tsitsiklis game each player i announced an offered payment $w_i^e \geq 0$ for each of the edges e on its paths (let $w_i^e = 0$ along edges not in the paths). For each edge e we allocate the bandwidth proportionally $x_i^e = b_e w_i^e / \sum_j w_j^e$, and then each user i can sent $x_i = \min_{e \in P_i} x_i^e$ bandwidth. The users than have a benefit of $U_i(x_i) - \sum_e w_i^e$. We added special rules on allocating bandwidth on edges e with $\sum_e w_i^e = 0$, and showed what a Nash equilibrium is like in this game, and showed that the price of anarchy in this game is at most 4/3rd, that is the welfare $\sum_i U_i(x_i)$ at Nash it at least a 3/4th fraction of the maximum welfare possible.

In defining the Nash equilibrium in this game, we assumed that each player i optimizes his values w_i^e to maximize his personal benefit, given that all the other values w_j^e remain fixed. Here we will assume a bit more flexible users. When player i increases his value w_i^e, then he will get more of the bandwidth of edge e. Now consider another player whose path uses edge e. After ith change the other player j gets less bandwidth on edge e than on his other edges. So naturally, player j will want to reallocate his money to offer more on edge e and maybe less on other edges.

Now define the **Sum Bid** game, in which each player i will only announce a single value w_i. Then the network manager (via a possible a bit complicated method) allocates the money w_i offered by agent i along the edges of path P_i in a pay to make sure that $x_i^e = x_i$ for all edges $e \in P_i$, and allocates the users the resulting bandwidth x_i. For answering this question, you do not have to worry about how the network can do this.

Consider the network of a single paths with k unit capacity edges, and $k + 1$ players, where player i needs capacity on edge i and player the extra player (say player 0) needs capacity on all edges. Assume that the utility of player 0 is $U_0(x) = \gamma x$ for some $\gamma > 0$ and the utility of players $i \geq 1$ is $U_i(x) = x$.

(a) What is the optimal allocation of bandwidth (maximizing total user happiness $\sum_i U_i(x_i)$, and what allocation do we get via the Johari-Tsitsiklis game. What is the worst efficiency loss one can get with this game of $k + 1$ players by varying γ.

(b) Show that the sum-bid game defined above leads to an allocation of $x_0 = \gamma/(\gamma + 1)$ for the user of the whole path, and $x_i = 1/(1 + \gamma)$ for all other users. What is the worst efficiency loss one can get with this game of $k + 1$ players by varying γ.