CS 684 Algorithmic Game Theory	**Scribe: James Lenfestey**
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1 Introduction

Last time we looked at the facility location game and observed that it had the exceedingly nice property that its potential function coincided with the social cost. This meant that both a Nash exists, and that the optimum solution would be a Nash. Further, we demonstrated that the value of an arbitrary Nash would be no worse than half that of the optimum.

In this lecture we covered more of Vetta's paper and showed that the above results extended to a more general class called *basic utility games.* In proving these results, we will have distilled away the key aspects of the facility location game. So the result may be more clear despite the increase in generality.

2 General framework

2.1 Definitions

Assume that we have a set of n players indexed by the variable k. Each player has a finite set of available actions $\mathcal{A}_k$. Then the set of all possible actions in the universe if $\mathcal{A} = \cup_{k=0}^{n} \mathcal{A}_k$.

We assume that the social value of a game configuration can be expressed as a function $V : 2^{\mathcal{A}} \to \mathbb{R}$. That is, value is defined on subsets $S \subseteq \mathcal{A}$, even if S contains more than one element from any of the $\mathcal{A}_k$'s.

Because we are trying to set up a game, we must also establish the value of configurations to the individual players (so that we know how they will act). If we take $S = \{a_1, ..., a_n | a_k \in \mathcal{A}_k\}$, configurations with exactly 1 action per player, then define

$$\alpha_i(S) = \text{value of } S \text{ to player } i.$$

In the context of our facility location game, the action set A_k was a set of possible locations for facility k. The set S consisted of one location for each facility. We saw that the social value was just the total value of the markets minus the transportation costs. Prices canceled out, and served only to apportion the available utility among the markets and the players. In our notation:

$$V(S) = \sum_{i \in markets} \pi_i - \lambda_{i\sigma(i)}, \quad \sigma(i) = \text{facility closest to market } i$$

The value to a player k:

$$\alpha_k(S) = \sum_i \rho_{ik} - \lambda_{ik},$$

where the index i ranges over the markets supplied by k, i.e. those closer to k than any other facility. λ_{ik} is the cost to transport goods from facility k to market i, and ρ_{ik} is the price facility k charges market i.

2.2 Assumptions

Our main result will apply to a class of games called *basic utility games*, of which our facility location game was an example. To be a basic utility game V and $\{\alpha_k\}$ must satisfy 3 properties:

(1) Value should be submodular (decreasing marginal utility), which means that for all sets $A \subseteq B \subseteq \mathcal{A}$ and $j \notin B$:

$$V(B+j) - V(B) \leq V(A+j) - V(a),$$

i.e. opening the second bookstore was not as beneficial as opening the first. The overloaded $+$ is taken to mean the union of a set and a singleton.

(2) $\sum_k \alpha_k(S) \leq V(S)$. The total value is no less than the sum of values for all of the players.

(3) $\alpha_k(S) = V(S) - V(S - a_k)$. The value player k contributes to the social value should be apportioned to player k.

If we replace property (3) with the slightly weaker (3'), then we get a *utility game*:

(3') $\alpha_k(S) \geq V(S) - V(S - a_k)$. Player k gets at least as much value as it contributes to the social value.

3 Results

Theorem 1 *In a basic utility game a deterministic Nash exists and the optimum solution is a Nash.*

Proof. (3) implies the $-V$ is a potential function. That is, if k changes action from a_k to a'_k, then its change in value will be $\alpha(S') - \alpha(S) = V(S - a_k + a'_k) - V(S - a_k) - V(S) + V(S - a_k) = V(S') - V(S)$. Since the space of configurations is finite, V can be maximized. This configuration will be a Nash of maximum social value. ■

To prove our next theorem, we require that V be monotone, $A \subseteq B \implies V(A) \leq V(B)$. This is less natural than the requirements above; in fact a facility location game with fixed costs might not satisfy this requirement.

Theorem 2 *The value of a Nash equilibrium in a utility game with monotone V is no worse than half optimal.*

Let $S = \{s_1, ..., s_n\}$ be a Nash equilibrium and $O = \{o_1, ..., o_n\}$ be the optimal configuration, the one that maximizes V. As S is a Nash, we know

$$\alpha_k(S - s_k + o_k) \leq a_k(S),$$

for every player k. If this were not true, then player k would strictly prefer o_k to s_k which would not be the case in a Nash.

Now let $O^k = \{o_1, ..., o_k\}$. As V is monotone:

$$
\begin{aligned}
V(O) &\leq V(O \cup S) && V \text{ monotone} \\
&= \sum_{k=1}^{n} [V(S \cup O^k) - V(S \cup O^{k-1})] + V(S) && \text{an identity} \\
&\leq \sum_{k=1}^{n} [V(S - s_k + o_k) - V(S - s_k)] + V(S) && \text{Property (1)} \\
&\leq \sum_{k=1}^{n} \alpha_k(S - s_k + o_k) + V(S) && \text{Property (3')} \\
&\leq \sum_{k=1}^{n} \alpha_k(S) + V(S) && S \text{ is Nash, noted above} \\
&\leq 2V(S) && \text{Property (2)}
\end{aligned}
$$

Thus, $V(S) \geq \frac{1}{2} V(O)$. ■

4 Comments

In class we proved Theorem 2 for basic utility games, when in fact (3') sufficed and the result held for more general utility games. Theorem 1, however, was only shown to hold for basic utility games.