CS 684 Algorithmic Game Theory	**Scribe: Yunsong Guo**
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1 Price of anarchy in discrete routing

Recall our setup for routing games. We have the delay on an edge e, ℓ_e(x), which is monotone increasing on x, and users select paths from s_i to t_i to route $dem(i)$ flow on the selected paths. In today's lecture we will discuss a type of discrete routing games, i.e., ∀ i user i has atomic and thus inseparable $dem(i) = 1$.

We have shown that for a non-atomic network routing game with only two edges, where one edge has constant delay 1 and the other edge has a delay function that is a degree d polynomial of the flow on that edge as in figure 1, the price of anarchy with demand equals to 1 can be as bad as $O(\frac{d}{\log d})$. In general, for polynomials of degree d delay functions with no restrictions on the demand of each user, the price of anarchy is bounded by $O(2^d d^{d+1})$, and is at least $\Omega(d^{d/2})$. Today we will analyze the discrete routing game with each delay function ℓ(x) to be a linear function on x, with non-negative coefficients, and show that the price of anarchy is approximately 2.61, which is ϕ^2 or ϕ+1 where ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

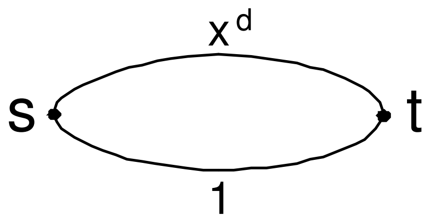

Figure 1: Example of polynomial delay of degree d

2 Discrete routing problem with unit demand and linear delay functions

We assume that all delay functions are linear and with non-negative coefficients , i.e.,

$$l_e(x) = a_e.x + b_e \ \ a_e, b_e \geq 0 \ \forall e \in E$$

In addition, we have $\forall i, dem(i) = 1$. As a result, this discrete routing game is a potential game, with the potential function to be $\phi = \sum_{e \in E} \sum_{i=1}^{x_e} l_e(i)$ where x_e is the number of paths on edge e. Hence, we know that there exists some deterministic Nash solution.

Consider in a Nash solution, x_e paths use edge e, and user i takes path P_i. In the optimal solution (where the total delay is minimized) x_e^* paths use edge e and user i takes path Q_i as illustrated in figure 2.

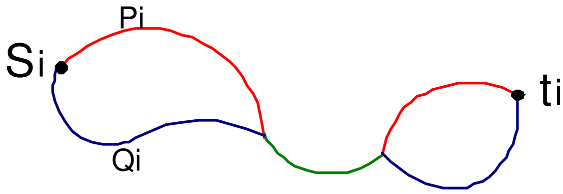

Figure 2: Paths P_i and Q_i may intersect

We can easily observe the fact that in the Nash solution, user i prefers path P_i to path Q_i for all i, so:

$$\ell_{P_i}(x) = \sum_{e \in P_i} \ell_e(x_e) \leq \sum_{e \in Q_i \setminus P_i} \ell_e(x_e+1) \; + \sum_{e \in Q_i \cap P_i} \ell_e(x_e)$$
$$\leq \sum_{e \in Q_i} \ell_e(x_e+1)$$

The first inequality is true because the flow on an edge $e \in Q_i \cap P_i$ will not change after user i switches from path P_i to Q_i. Now we are ready to prove today's main theorem.

Theorem 1 *In a discrete network routing game where each user has a unit demand and the delay on each edge is a linear function of the flow with non-negative coefficients, the price of anarchy is at most ϕ^2 where $\phi = \frac{1+\sqrt{5}}{2}$*

Proof: Let C(x) and C(x^*) denote the total delay of the Nash solution and optimal solution.

$$\begin{aligned} C(x) &= \sum_i \ell_{P_i}(x) \\ &= \sum_i \sum_{e \in P_i} \ell_e(x_e) \\ &\leq \sum_i \sum_{e \in Q_i} \ell_e(x_e+1) \\ &= \sum_{e \in E} x_e^* \ell_e(x_e+1) \\ &= \sum_{e \in E} x_e^*(a_e(x_e+1)+b_e) \\ &= \sum_{e \in E} [x_e^* a_e x_e + a_e x_e^* + b_e x_e^*] \\ &= \sum_{e \in E} x_e^* a_e x_e + \sum_{e \in E} (a_e x_e^* + b_e x_e^*) \qquad (1) \end{aligned}$$

Here $\sum_{e \in E} x_e^* a_e x_e$ is a mixed term, with the flow amounts from both the Nash and optimal solution on edge e. Also, $\sum_{e \in E}(a_e x_e^* + b_e x_e^*) \leq \sum_{e \in E} x_e^* \ell_e(x_e^*) = C(x_e^*)$ which is the total delay of the optimal solution. Recall the Cauchy-Schwarz inequality:

$$(\sum_{i=1}^n \alpha_i \beta_i)^2 \leq (\sum_{i=1}^n \alpha_i^2)(\sum_{i=1}^n \beta_i^2)$$

Let $\alpha_e = \sqrt{a_e} x_e$ and $\beta_e = \sqrt{a_e} x_e^*$, we have:

$$\sum_{e \in E} x_e^* a_e x_e \leq \sqrt{\sum_e a_e (x_e)^2 * \sum_e a_e (x_e^*)^2}$$

In addition, $\sum_e a_e (x_e)^2 \leq$C(x)= $\sum_e x_e (a_e x_e + b_e)$ and $\sum_e a_e (x_e^*)^2 \leq$C($x^*$)= $\sum_e x_e^* (a_e x_e^* + b_e)$. Hence (1) implies:

$$\begin{aligned} C(x) &\leq \sum_{e \in E} x_e^* a_e x_e + \sum_{e \in E} (a_e x_e^* + b_e x_e^*) \\ &\leq \sum_{e \in E} x_e^* a_e x_e + C(x^*) \\ &\leq \sqrt{C(x) C(x^*)} + C(x^*) \end{aligned}$$

Dividing both sides by $C(x^*)$, we have:

$$\frac{C(x)}{C(x^*)} \leq \sqrt{\frac{C(x)}{C(x^*)}} + 1$$

Recall that the golden ratio ϕ is the positive root of the equation $x^2 - x - 1 = 0$. Thus it is clear that $\frac{C(x)}{C(x^*)} \leq \phi + 1 = \phi^2$. ■