CS 684 Algorithmic Game Theory | **Scribe: Greg Studer**
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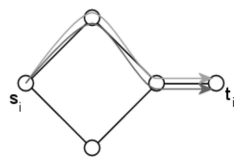
1 Discrete Routing Games

For the last week, we have been primarily concerned with non-atomic routing games, in which networks transport continuous amounts of flow. This lecture returns to the discrete routing games which were analyzed in earlier models. As before, the discrete routing model assumes a graph $G = (V, E)$, in which a user i chooses a single path from start node s_i to end node t_i. The delay function is unchanged, $l_e(x_e)$ for edge e, where x_e is the number of users using edge e.



The path a single user selects betwceen $s_i \to t_i$ is denoted P_i. Implicitly the users are assumed to be of equal size. The amount of delay on path P_i is given by:

$$\sum_{e \in P_i} l_e(x_e)$$

This model may apply more directly than the continuous version to certain real-world examples in which flow is discrete, such as the internet.

2 Nash Equilibria

Theorem: *$\exists$a set of paths P_i, $i = 1, 2, \ldots, k$ which is a Nash equilibrium.*

Claim: As in the continuous model, we can create a potential function

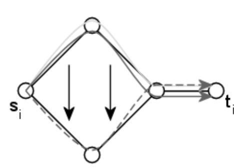
$$\phi = \sum_{e \in E} \sum_{i=1}^{x_e} l_e(i)$$

which is the discrete version of the function $\int_0^{x_e} l_e(\xi) d\xi$ seen in the continuous model. If there are no users on an edge, summing from 1 to x_e the summation above becomes a bit awkward, so we define $\sum_{i=1}^{x_e} l_e(i) = 0$ when $x_e = 0$.

Proof:

If ϕ is a potential function, then when a user i's experience changes by moving from edge P_i to Q_i, $Q_i \neq P_i$, this change is reflected in ϕ as well. More formally,

$$\triangle delay_i = \triangle \phi$$



By inspection of the graph, it is apparent that the change in potential function for a user i switching from P_i to Q_i is

$$\triangle\phi = \sum_{e\in(P_i-Q_i)} l_e(x_e) - \sum_{e\in(Q_i-P_i)} l_e(x_e+1)$$

where x_e is the number of users on the edge prior to the switch. This is the change in delay that user i experiences as well, because users only see changes local to their own current path.

Discussion:

Since we know now that ϕ is a potential function, there are two scenarios which can occur when optimizing ϕ. We may be able to reduce ϕ infinitely, in which case a Nash equilibrium does not exist. On the other hand, if we are not able to optimize ϕ past some value N, the paths for all users when $\phi = N$ form a Nash equilibrium. Because we are using a finite model with discrete user loads, it is impossible to reduce ϕ infinitely, and so the equilibrium must exist.

We have yet to say anything about the quality of the Nash, only that it exists. In the next proof, we will attempt to show that the Nash is a high-quality flow, but to do so, we need some notion of what quality actually means. For our purposes, the quality of a solution is the inverse of the cost of the solution, where the cost C is:

$$C = \sum delays = \sum_{e\in E} x_e l_e(x_e) = \sum_{i\in users} \sum_{e\in P_i} l_e(x_e)$$

To determine the quality of the Nash solution then, we can compare the Nash cost to the cost of an optimal solution. Comparing the cost of an optimal solution to a Nash solution where users act selfishly is actually a very general concept in game theory called the "Price of Anarchy." In our discrete network flow problem we may have multiple Nash solutions to choose from, but will assume the magical power of being able to choose the initial flow in our network. We would prefer a stable flow in which selfish users do not want to switch paths, otherwise our total delay cost will fluctuate as users try to change their routing. This implies we will choose the best Nash solution we can, and the penalty in delay cost we may incur because the optimal solution is not Nash is called the "Price of Stability." The theorem below makes this idea more rigorous.

3 Price of Stability

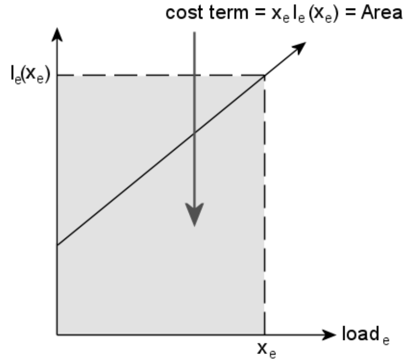
Theorem: *If the potential ϕ and the cost C satisfy $\phi \le C \le \alpha\phi$ for all solutions ($\alpha > 1$), then the Nash equilibrium minimizing the potential ϕ has a cost $\le \alpha\times$the minimum possible cost.*

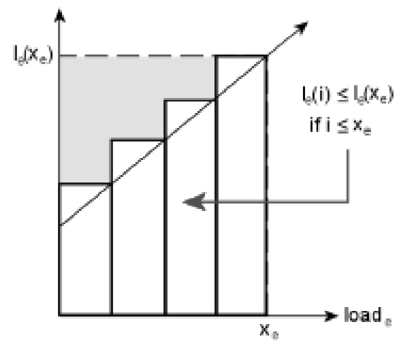
To motivate this theorem, first we will demonstrate an example problem in which such an α exists.

Example: A network with linear delays and monotone load function $l_e(x_e) = ax_e + b$, $a, b \ge 0$. By definition:

$$C = \textstyle\sum_{e\in E} x_e l_e(x_e) \quad \phi = \sum_{e\in E} \sum_{i=1}^{x_e} l_e(i)$$

On a single edge,





As can be observed from the above graphs, if the delay is monotone, then $\phi \leq C$. In fact, this statement actually holds for every monotone delay function l_e. In the worst case where $b = 0$, $C \leq 2\phi$, so $\alpha = 2$. Therefore, if the delay is linear, there is always a Nash equilibrium with twice the cost of the optimum solution.

Proof:

To prove the above theorem, we have to compare the particular Nash minimizing ϕ to the cost of the optimal flow. To help us visualize this process, we can create a table comparing ϕ and C on the network edges. (x_e is the user load under the Nash flow, while x_e^* is the user load under optimum flow.)

$$\begin{array}{ccc} \phi(x_e) & versus & \phi(x_e^*) \\ C(x_e) & versus & C(x_e^*) \end{array}$$

By choice of x_e, $\phi(x_e) \leq \phi(x_e^*)$, and as we have seen in the above example, $\phi \leq C$. Therefore:

$$\phi(x_e) \leq \phi(x_e^*) \leq C(x_e^*)$$

By assumption, $\alpha\phi(x_e) \geq C(x_e)$ or:

$$\phi(x_e) \geq \frac{1}{\alpha} C(x_e)$$

Chaining the inqualities together, we then have:

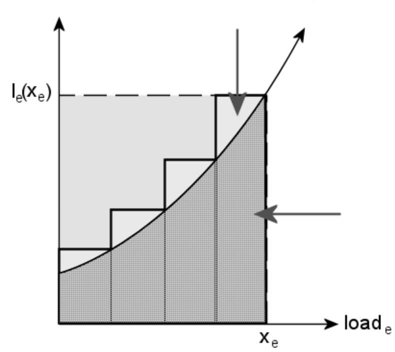
$$\frac{1}{\alpha} C(x_e) \leq \phi(x_e^*) \leq C(x_e^*)$$

$$C(x_e) \leq \alpha C(x_e^*)$$

This is what we wanted to prove, that the total cost of the Nash flow is $\leq \alpha \times$the cost of the optimum flow.

To demonstrate how this theorem can be applied, we analyze a second example where the delay function is a polynomial of degree d. Formally...

Example: *In a network where $l_e(x_e) = a_d x_e^d + a_{d-1} x_e^{d-1} + \cdots + a_0$ is a polynomial function of degree d, with all coefficients $a_i \geq 0$, the price of stability is $(d+1)\times$cost of the optimal solution.*



We need to compare the integral and sum areas

As before, because l_e is monotone, $\phi \leq C$. The area corresponding to the potential function x_e is greater than the integral of the load function over all x_e.

$$\phi_e = \sum_{i=1}^{x_e} l_e(x_e) = \sum_{j=0}^{d} a_j x_e^j \geq \int_0^{x_e} l_e(\xi) d\xi$$

The cost term C_e on the edge is:

$$C_e = x_e l_e(x_e) = x_e \sum_{j=0}^{d} a_j x_e^j$$

Taking the minimum cost of the edge to be the value of the integral and evaluating yields:

$$\phi_e \geq \sum_{j=0}^{d} a_j (\frac{1}{j+1}) x_e^{j+1}$$

$$C_e = \sum_{j=0}^{d} a_j x_e^{j+1}$$

So comparing termwise in the above two formulas, it is apparent that $\phi \leq C \leq (d+1)\phi$. Applying the theorem above, we can then conclude that:

$$C_{min} \geq \frac{1}{d+1} C_{Nash_{min}}$$

With a monotone polynomial delay function of degree $\leq d$, the price of stability is $(d+1) = \alpha$. In other words, there must exist a Nash flow with a social cost $\leq (d+1)\times$(minimum cost of any solution). Compare this answer to the social cost of $\frac{d}{log(d)}$ in non-atomic games.