**Lecture Notes Dec. 2 Bistra Dilkina**

# 1 Project

Again the project should cover 1-2 papers. The project should summarize the findings of the paper. It should present some interesting result(s) from the paper in a format similar to the class lecture notes. You should also over a criticism (positive or negative) of the papers. It is ok to copy definitions and equations. No lower or upper bound on pages, but around 7 should be right. Office hours next week are as usual: Tuesday and Friday.

# 2 Truthful Games

Truthful games are quite different from Nash games and on Monday we will compare the trade off between using either. Today we will talk about truthful games and analyze how well we can do with respect to making money.

The simplest framework is selling one kind of good which has unlimited copies (obtainable for free by the seller). For example, software. There are $n$ players with valuations $v_i$ for having the good. WLOG assume $v_1 \geq v_2 \geq ... \geq v_n$.

An upper bound to the profit the seller can make is $\sum_{i=1}^{n} v_i$, by selling the good to each player at price equal to the player's valuation. But doing this will give an incentive to players to lie about their valuations.

To preserve truthfulness, we have two obvious options:

1) Sell only one copy: Sell to the player with highest valuation, $v_1$, at price equal to the valuation of the second highest player valuation, $v_2$ (Vickery Auction). This option is really bad when $v_1 >> v_2$. But it is hard to do anything else and still preserve truthfulness.

2) Sell $k$ copies: Sell to the $k$ players with highest valuations, $v_1, .., v_k$ and charge each one of them $v_{k+1}$ (extending Vickery Auction in the obvious way).

How do we choose $k$?

Jason Hartline, Goldberg and Karlin have papers that address this issue. In what follows we discuss some of their results.

The maximum income that we can get by charging any fixed price to all players that buy a good is $\max_{k \geq 2} k v_k$, where $k$ is the number of players that buy at that price. We avoid setting $k$ to 1 to avoid the problem with option 1.

# 3 Goldberg and Hartline (Negative result)

In an auction, every player $i$ is asked a price $p_i$, which is the threshold of $i$ (as discussed last time but then $p_i$ was called $w_i$). In a **symmetric auction** $p_i$ depends only on the valuations $v_1, ..v_n$ and not on player's $i$ order in the valuation list, i.e. $p_i$ is name-independent.

**Theorem 3.1** *No symmetric auction mechanism can approximate* $\max_{k \geq 2} k v_k$ *to any constant factor.*

Proof omitted. However, the authors show that there exists a truthful mechanism that does better, but the $p_i$'s depend both on the valuations and on the name of the player $i$.

# 4 Randomized Auction

We will show a mechanisms where the expected income is 1/4 of $\max_{k \geq 2} k v_k$.

Randomly and independently with probability 1/2 we put each player in one of two bins (Red and Blue bin). We order all players in the Red bin, say there are $k_r$ of them, by valuations where the valuations are $r_1 \geq .. \geq r_{k_r}$. Similarly the $k_b$ players in Blue bin are ordered by their valuations $b_1 \geq ... \geq b_{k_b}$. We compute: $V_R = \max_{k \geq 1} k r_k$ and $V_B = \max_{k \geq 1} k b_k$.

Now, we ask of the players in the Red bin an income of $V_B$, and from the players in the Blue bin an income of $V_R$. If one of the bins says yes (they can provide such total income) , they get the good, otherwise no.

**Claim 4.1** *If $V_R \leq V_B \Rightarrow$ the method will make income $V_R$ from the Blue bin.*

Note that if $V_R = V_B$, then both bins answer yes and the total income is $V_R + V_B$. However, the probability of $V_R = V_B$ is 0 given the way we selected the players in the bins.

**Lemma 4.1** *Given any $C \leq \max_{k\geq 1} kb_k = V_B$, a truthful auction with the players of the Blue bin can extract $C$ income.*

We give the mechanism that extracts this income:
- start with $k = k_b$
- until $k > 0$
- offer price $p = C/k$
- delete all players that cannot pay $p$, i.e. $v_i < p$
- if there were no such players to delete, sell to all $k$ players one good at price $p$
- otherwise set $k =$ the number of remaining players and repeat

**Claim 4.2** *This method is truthful.*

**Proof** The player has no incentive to lie and say that they can pay, when $v_i < p$, because in the following iterations of the method the price will only go up and there is no way that he will get a chance to buy. Hence, each player will truthfully leave when the price is too high, and the other players will remain hoping to buy a good.

**Claim 4.3** *If $C \leq \max_{k\geq 1} kb_k = V_B$, then the method will always manage to sell to some players and make income of $C$.*

**Proof** The method always offers price such that $pk = C$, where $k$ is the number of remaining players. In the case when we sell, no player of the $k$ remaining players rejects the price and hence we make exactly $C$ income. The only other case is when all players are deleted and the method makes no money but it cannot happen. We are guaranteed to succeed at some iteration if $\exists k$ such that $p = C/k \leq b_i,\ forall i \in [1, k]$. But since $C \leq \max_{k\geq 1} kb_k = V_B$ and $b_1 \geq ... \geq b_{k_b} \Rightarrow$ such $k$ always exists, i.e. $k = arg \max \max_{k\geq 1} kb_k$.

The method above will not be truthful if $C$ depends on the valuation of any player in the Blue bin, because the player will have incentive to lie and thus influence the value of $C$ in his advantage. So we offer to the players in the Blue bin $C = V_R$ that depends only n players in the Red bin. Notice that the whole argument is symmetric with respect to Red and Blue bin, when $V_B \leq V_R$.

There is price discrimination since only players from one of the bins (Red or Blue) gets to buy goods and at uniform price.

# 5 How much do we make with the Randomized Auction?

Let $k = arg \max_{k\geq 2} kv_k$ and $V_{max} = \max_{k\geq 2} kv_k$.

1) If k=2, with probability 1/2 in our method both top players are in one bin(say Blue) and we make $V_R << V_B$. Otherwise with probability 1/2 they are in the different bins. WLOG let $v_1$ be in Red bin, and $v_2$ be in Blue bin, then $V_R \geq v_1$ and $V_B \geq v_2$. So we make at least $v_2$. Hence, $E[income] \geq v_2/2 = 2v_2/4 = V_{max}/4$.

2) If k=3, with probability 3/4, the top 3 players are distributed 1 to 2 in the two bins. In the worst case $v_3$ is in one bin, and $v_1$ and $v_2$ are in the other bin. Then the income in this case is at least $v_3$. So $E[income] \geq 3v_3/4 = V_{max}/4$.

**Claim 5.1** *The randomized auction achieves expected income that is at least $1/4$ of the maximum possible income.*

For higher $k$, the expected income is even better than 1/4 of the maximum income.