CS 684 Algorithmic Game Theory	Scribe: Greg Studer
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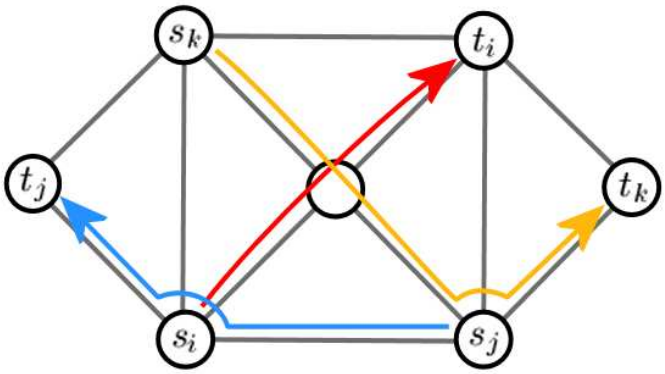
This week's lecture is a continuation of the truthful systems discussion from last week. This time, as last time, we consider a simple truthful system, in which each agent has only one parameter to report.

1 Example Game



This example takes place on a graph G, where each agent wants a path in G from nodes $s \to t$ because the path has a value v. Agents 1 to k have paths from $s_i \to t_i$, with value v_i. The goal in this game is to find a subset $I \subseteq \{1 \dots k\}$ where each agent $i \in I$ has a disjoint path P_i in G, while maximizing the social welfare $\sum_{i \in I} v_i$. For simplicity, disjoint paths are assumed, but other variants of this game exist where a certain number of paths may overlap on an edge. The objective here is welfare maximization, but other games are possible using other measures, such as income maximization.

We desire a truthful game for the usual reasons: simplicity and predicability. Note that each player i currently has three parameters s_i, t_i, and v_i. It is possible that a user might lie about different combinations of these parameters, but in this example, we assume that the s_i and t_i are public knowledge.

2 The Path Selection Algorithm

In this system, agents only report v_i, and then some public algorithm selects I, the paths P_i for $i \in I$, and the payments w_i for each of the paths. We only collect this payment w_i so that agents are truthful. The total benefit to agent i recieving a path is $v_i - w_i$. Last week, using the VCG mechanism, it was shown that if welfare maximization is computable, a truthful system can be created. Here, unfortunately, welfare maximization on disjoint paths is NP-hard. What else can we do?

This problem can be broken into two major parts:

- Select the agent paths
- Select the agent payments

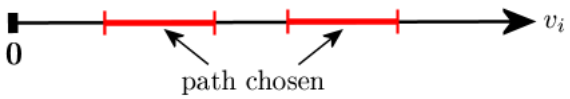
The path selection algorithm should be good at both maximizing welfare and maintaining truthfulness. We introduce:

Property: If v_i is high enough, with all other $v_j \neq v_i$ fixed, agent i should be able to ensure they are included in I.

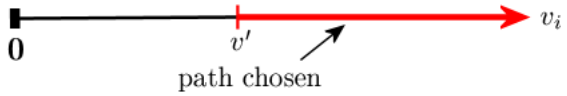
Basically, this property ensures that the values of the agents are not ignored, otherwise a simple truthful algorithm would simply be to ignore the agents' values entirely. While this property does not ensure welfare is maximized, it does ensure that high values are respected.

Property: For a given v_i where $v_j \neq v_i$ are fixed, $\exists v'$ such that if $v_i > v'$, $i \in I$, and if $v_i < v'$, $i \notin I$. Note the strict inequality.

Graphically, in a system where this property does not hold,

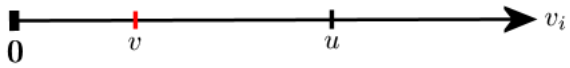


The separate regions in the v_i number line rules out truthfulness, because an agent in a higher v_i region could reduce their value and still get their path. Instead, v_i needs to look like...



This property is known as monotonicity. Does it necessarily lead to truthfulness?

Proof: (by contradiction) Assume $v_i$is not monotonic, with some $u > v$ where at $v_i = u$, $i \notin I$ and $v_i = v$, $i \in I$. Then the algorithm cannot be truthful.



At $v_i = v$, where $i \in I$, path P_i would require a payment w_i. $w_i \leq v_i$, because otherwise $v_i - w_i < 0$.

But then at $v_i = u$, where $i \notin I$, agent i would want to lie and say their $v_i = v$, because their benefit at u would be 0. This would be less than their benefit $v_i - w_i \geq 0$ at v.

Therefore, every truthful system has this monotonicity property. It remains to be proven that any monotonic algorithm has payments w_i that make it truthful.

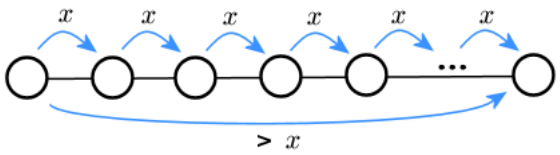
Proof: In a monotonic system, every person has cutoff v'. If we make payments for a path P_i equal to v', the system will be truthful, because if an agent's value is any amount above v', it won't matter to them, and if the value is under v', the agent will only want to increase their price if their true value v_i is greater than v'.

Can we construct an algorithm with these two properties? Our goal is to maximize social welfare $\sum_{i \in I} v_i$. (It is easy to see that a function finding the true maximum will be monotone for each v_i.) Since this function will be NP-complete, we want a good approximation algorithm instead that is monotone. The disjoint union of paths has only bad approximation algorithms, but fortunately, these simple algorithms tend to be monotone.

3 Approximation Algorithms

Example: A greedy path search. Sort v_i by size, then assign paths for the agents with largest v_i first.

This simple greedy algorithm satisfies the first property, because an agent can always increase its value enough to ensure its path is chosen. The algorithm is also monotone, because if an agent has a path and increases its value, their path will still always be chosen (assuming no one else changes). Because it supports these properties, the greedy algorithm leads to a truthful system. However, the greedy algorithm has bad worst-case bounds, with an approximation factor of about $n\times$ the optimal value, where n is the number of nodes in the graph. This bad case is illustrated below:

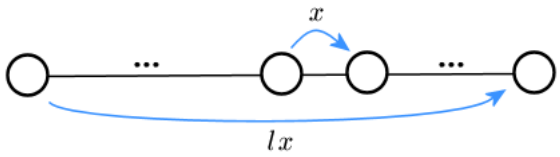


Example: Modified greedy path search. Sort all the unassigned agents (agent $i \notin I$) by $\frac{v_i}{f(l_i)}$, where l_i is the length of the current shortest path from $s_i \to t_i$. This length is recalculated after each agent path is chosen.

This function satisfies the first property, because an agent that increases its value enough will still always be able to ensure their path (though if their path is long, the value must increase more). The algorithm is also monotone, because if an agent has a path and increases its value, the $f(l_i)$ remains constant, and so it will always sort the same or higher in the list of agents. As above, the modified greedy algorithm is therefore truthful. This modified algorithm still has bad worst case bounds:

- if $f(x) = 1$, then the approximation factor is again n.
- if $f(x) \geq x$, the approximation factor is $> n$.

This can be seen in the diagram below:



If these are the extremes, then a tradeoff between 1 and x would yield a better approximation factor. Though not proven here, if $f(x) = \sqrt{x}$, this gives a $\sqrt{n} + 1$ approximation factor. It is possible to do better, if more than one path is allowed on a particular edge of G, but this also requires a more complex approximation function.