CS 684 Algorithmic Game Theory	**October 12, 2005 (Wednesday)**
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1 Overview of today's class

In the last class, we were discussing Johari-Tsitsiklis game for a network with single link and derived Price of Anarchy bound of 3/4. Today we will continue looking at Johari-Tsitsiklis game, extending it to general networks and we will also extend the Price of Anarchy result to this case. The basic idea involved is to reduce the general network case to a bunch of independent single link cases and to use the bound from previous class.

2 Bandwidth sharing in networks—the model

We first discuss the model for bandwidth sharing in networks (continuing the discussion from last lecture).

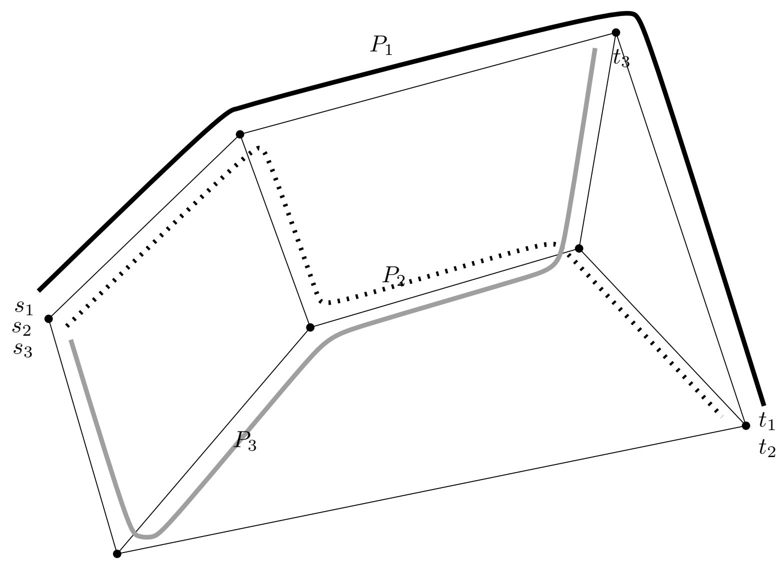

Figure 1: An example network. Each user has a path from its source to its sink.

There is a graph $G = (V, E)$ and bandwidth limit b_e on each edge $e \in E$. There are also k users, each having a source and a sink, called s_i and t_i, and a path from s_i to t_i called P_i. Note that the path P_i is fixed for the user i and she does not have any control over choosing a different path (see Figure 1 for an example). Each user also has a utility function $u_i(x)$ which is a function of amount of traffic rate she can push through her path P_i (subject to bandwidth constraints). Function $u_i(x)$ is assumed to be (strictly) monotone increasing, continuously differentiable, and strictly concave. User i wants to send flow x_i along her path to maximize her utility (minus the total money paid). A socially optimum solution maximizes the aggregate utility of all the users.

2.1 Possible variants of the model

The above model can be changed a little bit in many different ways, for example allowing users to select a path or allowing them to split their flow. We mention both the cases below.

Allow users to select their path In this case, pure (or deterministic) Nash equilibrium may not exist. Indeed, there are examples which will make the users oscillate among many paths, never reaching an equilibrium.

Allow users to split their traffic In this case, Nash equilibria exist and the same analysis goes through, only making the algebra messier. We will not consider this case either.

We will stick to the model mentioned in Section 2.

3 Optimality conditions from last time

An optimum solution is a solution to the following optimization problem.

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} u_i(x_i) && \text{(OPT)} \\ \text{s.t.} \quad & \sum_{i:e \in P_i} x_i \le b_e \qquad \forall e \in E \\ & x_i \ge 0. \end{aligned}$$

Last time we saw the conditions when the traffic amounts $x_1^*, x_2^*, \ldots, x_n^*$ are optimum solution to the above optimization problem. Karush-Kuhn-Tucker Theorem leads to the following conditions (also mentioned in the last lecture) for a solution to be an optimum solution for OPT.

Condition 1 (Karush-Kuhn-Tucker) *A solution $x_1^*, x_2^*, \ldots, x_n^*$ is an optimum solution to OPT if and only if*

$$\sum_{i:e \in P_i} x_i^* \le b_e \quad \forall e \in E \qquad \text{(Feasibility)}$$

$$x_i^* \ge 0 \quad \forall i = 1, 2, \ldots, n \qquad \text{(Feasibility)}$$

$$\exists p_e^* (e \in E) \quad s.t. \quad p_e^* > 0 \implies \sum_{i:e \in P_i} x_i^* = b_e \quad \forall e \in E \qquad \text{(Dual Comp. Slackness)}$$

$$x_i^* > 0 \implies u_i'(x_i^*) = \sum_{e:e \in P_i} p_e^* \quad \forall i = 1, 2, \ldots, n \qquad \text{(Primal Comp. Slackness)}$$

$$x_i^* = 0 \implies u_i'(x_i^*) < \sum_{e:e \in P_i} p_e^* \quad \forall i = 1, 2, \ldots, n.$$

Also note that Equation (Primal Comp. Slackness) says that the solution x_i^* is optimal for user i for given "prices" $p_e^*, e \in E$. This can be seen by the following. For given prices $p_e^*, e \in E$, user i's payoff is $u_i(x_i) - \sum_{e \in P_i} (x_i \cdot p_e^*)$. To maximize this, we differentiate it (with respect to x_i) and equate it to zero, resulting in Equation (Primal Comp. Slackness).

4 The game

We now define the game which the users play on the above mentioned network. The steps are as follows.

Bid User i offers w_i^e for edges $e \in P_i$. She does not offer anything for $e \notin P_i$ or equivalently $w_i^e = 0$ for $e \notin P_i$. This is done by all users $i = 1, 2, \ldots, n$.

Bandwidth allocation User i gets

$$x_i^e = \frac{w_i^e}{\sum_j w_j^e} \cdot b_e \tag{1}$$

amount of bandwidth on edge e. The sum in the denominator above is over all j who offered something on the edge e. This can also be taken over all users j because for the users who did not offer anything for edge e, $w_j^e = 0$.

An alternative way to look at bandwidth allocation is the following. Once all the users bid money on edges, prices p_e (dollars per unit bandwidth) are implicitly set on each edge according to the following equation.

$$p_e = \frac{\sum_j w_j^e}{b_e}.$$

The users are now assigned bandwidth $x_i^e = \frac{w_i^e}{p_e}$ on each edge, according to the price p_e.

There are still some caveats in the bandwidth distribution. See Section 4.1 for a discussion.

Bandwidth selection User i selects the maximum rate x_i at which she can send her traffic. This is given by

$$x_i = \min_{e \in P_i} x_i^e. \tag{2}$$

Payoff/value Payoff derived by user i in this process is

$$u_i(x_i) - \sum_{e \in P_i} w_i^e. \tag{Payoff}$$

The goal of each player is to maximize her payoff given by Equation (Payoff).

4.1 Caveats in bandwidth allocation

When bandwidth was allocated on every edge according to Equation (1), it was assumed that at least one user bids a positive amount on this edge (otherwise, the denominator becomes zero). What happens if none of the user bids a positive amount on this edge? For fixing this, the game has the following (extra) rules for bandwidth allocation.

1. While bidding amounts for edges, if a user i does not bid money on an edge e (that is $w_i^e = 0$), then she can specify an amount x_i^e that she wants for "free".

2. If any user bids money on edge e ($w_i^e > 0$ for some j), then usual bandwidth rule given by Equation (1) applies.

3. If $w_j^e = 0$ for all $j \in [n]$ (here $[n] = \{1, 2, \ldots, n\}$) and $\sum_j x_j^e \leq b_e$ (sum of "free" bandwidth asked for is at most the capacity), then each user j gets x_j^e amount of bandwidth for free[1].

4. If $w_j^e = 0$ for all $j \in [n]$ and $\sum_j x_j^e > b_e$ (sum of "free" bandwidth asked for is greater than the capacity), no user is given any bandwidth for free (on this edge).

5 Properties of a Nash equilibrium

We have defined the game and all of its rules. We now looks into the conditions that are necessary and sufficient for a Nash equilibrium.

In a Nash equilibrium, each user tries to maximize her own payoff (given by Equation (Payoff)) given other users' strategies. Equivalently, she want to bid money w_i^e on edges e in such a way so as to maximize

$$u_i(x_i) - \sum_{e \in P_i} w_i^e,$$

where $x_i = \min_{e \in P_i} x_i^e$. This function is a function of w_i^e for $e \in P_i$ and x_i depends on these variables in a complicated way (see Equations (1) and (2) for precise dependence). How do we maximize this?

An important observation is to concentrate on x_i instead of w_i^e for all edges $e \in P_i$. The user should have a target amount of bandwidth in mind and bid on edges accordingly. This observation gives rise to the following optimization problem for player i.

Given $\{w_j^e, j \neq i, e \in E\}$, find x_i solving the following

$$\max_{x_i} \quad u_i(x_i) - \sum_{e \in P_i} w_i^e \tag{3}$$

$$\text{subject to} \quad x_i = \frac{w_i^e}{\sum_{j \neq i} w_j^e + w_i^e} \cdot b_e \quad \forall e \in P_i. \tag{4}$$

This optimization problem is solved by taking derivative with respect to x_i and equating it to zero. As this involves lengthy and messy algebra, we will state the final result,

$$u_i'(x_i) = \sum_{e \in P_i} \frac{p_e}{1 - \frac{x_i}{b_e}}, \tag{5}$$

where p_e is given by $\frac{\sum_j w_j^e}{b_e}$. Now we can state the conditions for a solution to be a Nash equilibrium.

[1]In this case, we are not considering network owner's perspective. Why will the network owner provide bandwidth for free? Or even more generally, we are not considering network owner's perspective on any of the edges. The proportion division is NOT maximizing the network manager's income, even without the "free" bandwidth issue. For the whole game, we simply assumed a manager who just wants to do a good job for people, and not care about her income at all. This assumption is not realistic but for now we will assume it. If time permits, we might come back to this issue towards the end of the course.

Condition 2 *Let $x_1, x_2, \ldots, x_n$ be the traffic amounts sent by users along their paths. This configuration (or solution) is a Nash equilibrium if and only the following conditions hold.*

$$\sum_{i:e\in P_i} x_i \le b_e \quad \forall e \in E \qquad \text{(Feasibility)}$$

$$x_i \ge 0 \quad \forall i = 1,2,\ldots,n \qquad \text{(Feasibility)}$$

$$\exists p_e (e \in E) \quad s.t. \quad p_e > 0 \implies \sum_{e:e\in P_i} x_i = b_e \qquad \text{(Nash Dual Slackness)}$$

$$x_i > 0 \implies u_i'(x_i) = \sum_{e\in P_i} \frac{p_e}{1 - \frac{x_i}{b_e}} \qquad \forall i = 1,2,3,\ldots,n$$

(Nash Primal Slackness)

$$x_i = 0 \implies u_i'(0) < \sum_{e\in P_i} p_e \qquad \forall i = 1,2,3,\ldots,n.$$

5.1 Relation to one link network case

In one link case, we saw in last class that a flow configuration $x_1, \ldots, x_n$ is a Nash equilibrium if and only if (assuming $B = 1$)

$$x_i > 0 \implies u_i'(x_i)(1 - x_i) = p \qquad (6)$$

$$x_i = 0 \implies u_i'(0) < p, \qquad (7)$$

and some other feasibility requirements are satisfied. These equations are just a special case of Equation (Nash Primal Slackness). This is easily seen by putting $b_e = 1$ and removing the summation in Equation (Nash Primal Slackness) (as there is just one link). Indeed, using Equation (Nash Primal Slackness), Nash condition in the one link case can be extended to capacity B link by

$$x_i > 0 \implies u_i'(x_i)\left(1 - \frac{x_i}{B}\right) = p \qquad (8)$$

$$x_i = 0 \implies u_i'(0) < p. \qquad (9)$$

6 Price of Anarchy result

We state the final result as the following theorem.

Theorem 3 ([JT04]) *In the bandwidth allocation game described above, worst case ration between value of Nash and value of an optimum solution is at least $\frac{3}{4}$.*

Proof. We have (almost) seen all the pieces of proof already. We just need to patch them up. The idea is the following. We will modify the problem in which the ratio can get only worse and from there reduce the problem to single link case. We will use the 3/4 worst case ratio from last lecture to finish the proof.

Let us fix a Nash equilibrium having flow amount $x_1, x_2, \ldots, x_n$ for n users.

Define a new utility function for users as a function of flow on each of edge (not just a function of total flow),

$$H_i(y_i^e; e \in E) = \sum_{e \in P_i} \frac{p_e}{1 - \frac{x_i}{b_e}} (y_i^e - x_i) + u_i(x_i). \tag{10}$$

Note that $H_i(y_i)$ is a function of y_i^e only, as x_i are fixed by fixing the Nash equilibrium. Now we will make a series of claims, eventually proving the required bound of 3/4.

Claim 4 *Let $H_i(y_i^e; e \in E)$ be defined as above. Then,*

$$H_i \begin{pmatrix} y_i^e; & e \in P_i \\ 0; & e \notin P_i \end{pmatrix} \geq u_i\left(y_i\right), \tag{11}$$

where $y_i = \min_{e \in P_i} y_i^e$. Notation $\begin{pmatrix} y_i^e; & e \in P_i \\ 0; & e \notin P_i \end{pmatrix}$ represents a vector indexed by $e \in E$, the components corresponding to edges in the path P_i have value y_i^e, and others carry value 0.

Proof of Claim 4: The inequality is easily seen from the following chain:

$$\begin{aligned} H_i \begin{pmatrix} y_i^e; & e \in P_i \\ 0; & e \notin P_i \end{pmatrix} &= \sum_{e \in P_i} \frac{p_e}{1 - \frac{x_i}{b_e}} (y_i^e - x_i) + u_i(x_i) \qquad \text{(By definition)} \\ &\geq \sum_{e \in P_i} \frac{p_e}{1 - \frac{x_i}{b_e}} (y_i - x_i) + u_i(x_i) \qquad (H_i \text{ is monotone in each coordinate}) \\ &= u_i'(x_i)(y_i - x_i) + u_i(x_i) \qquad (\text{Condition on } u_i'(x_i) \text{ in Nash equilibrium}) \\ &\geq u_i(y_i) \qquad (\text{Concavity of } u_i(x) \text{ function}). \end{aligned}$$

The claim follows. **(End of Proof of Claim 4)**

In the game corresponding to functions H_i's as utility function, total utility is a functions of all y_i^e and it is separable (can be written as the sum of utilities of edges) and hence players play different game on different edges. What does a Nash equilibrium correspond to in this game? We will first clarify this point.

In the modified game with utility functions $H_i(y_i^e; e \in E)$, a strategy profile (one strategy for each player)

$$(y_1^e; e \in E, \quad y_2^e; e \in E, \ldots, \quad y_n^e; e \in E)$$

is a Nash equilibrium if it is a Nash equilibrium independently on each edge $e \in E$. This means that for every edge $e \in E$, strategy profile

$$(y_1^e, y_2^e, \ldots, y_n^e)$$

is a Nash equilibrium for edge e. This happens for all edges $e \in E$.

Claim 5 *The original Nash equilibrium $x_1, x_2, \ldots, x_n$ is a Nash equilibrium with respect to H_i payoffs as well.*

Proof of Claim 5: First of all, what do we mean by the statement of the claim. We actually mean to say that natural extension of strategy profile $x_1, x_2, \ldots, x_n$ is a Nash for H_i. By natural extension of $x_1, x_2, \ldots, x_n$, we mean the following:

$$\left(\begin{pmatrix} x_1; & e \in P_1 \\ 0; & e \notin P_1 \end{pmatrix}, \begin{pmatrix} x_2; & e \in P_2 \\ 0; & e \notin P_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n; & e \in P_n \\ 0; & e \notin P_n \end{pmatrix} \right) \tag{12}$$

To see the claim, note that the condition for the players using edge e to form a Nash equilibrium is the following (see Equation (8))

$$\left.\frac{\partial H_i}{\partial y_i^e}\right|_{y_i^e = x_i} \left(1 - \frac{x_i^e}{b_e}\right) = p_e.$$

Plugging in the definition of H_i from Equation (10), we see that the left hand side is

$$\frac{p_e}{1 - \frac{x_i}{b_e}} \cdot \left(1 - \frac{x_i^e}{b_e}\right)$$

which is equal to the right hand side (note that $x_i = x_i^e$ for $e \in P_i$). This proves the the statement claimed. **(End of Proof of Claim 5)**

We are now about the finish the proof. The optimum function value might have increased in the game with H_i's utility function (see Claim 4) and the old Nash is still a Nash (see Claim 5). Hence, the price of anarchy could possibly have gotten worse, it could not have been better. We will prove the 3/4 bound on the modified game and that will prove the 3/4 bound for the original game.

In the modified game, a player plays game on each of her P_i edges separately. The bound of 3/4 proved in last lecture applies to each of the edge separately, proving the theorem for the whole network. To be more precise, let y_i^e be a Nash for new game and y_i^{*e} be an optimum solution.

$$\begin{aligned}
\frac{\text{Nash(Old)}}{\text{Opt(Old)}} &\geq \frac{\text{Nash(New)}}{\text{Opt(New)}} \\
&= \frac{\sum_{i=1}^n H_i(y_i^e; e \in E)}{\sum_{i=1}^n H_i(y_i^{*e}; e \in E)} \\
&= \frac{\sum_{i=1}^n \left(u_i(x_i) + \sum_{e \in E} H_i^e(y_i^e)\right)}{\sum_{i=1}^n \left(u_i(x_i) + \sum_{e \in E} H_i^e(y_i^{*e})\right)} \qquad \left(\text{Let } H_i^e(y_e) = \frac{p_e}{1 - \frac{x_i}{b_e}}(y_i^e - x_i)\right) \\
&\geq \frac{\sum_{i=1}^n \left(\sum_{e \in E} H_i^e(y_i^e)\right)}{\sum_{i=1}^n \left(\sum_{e \in E} H_i^e(y_i^{*e})\right)} \\
&= \frac{\sum_{e \in E} \sum_{i: e \in P_i} H_i^e(y_i^e)}{\sum_{e \in E} \sum_{i: e \in P_i} H_i^e(y_i^{*e})} \qquad \text{(Interchanging the order of the summation)} \\
&\geq \min_{e \in E} \frac{\sum_{i: e \in P_i} H_i^e(y_i^e)}{\sum_{i: e \in P_i} H_i^e(y_i^{*e})} \\
&\geq \frac{3}{4} \qquad \text{(Single link bound from last class).}
\end{aligned}$$

This proves Theorem 3. ■

7 The coming attractions

The bandwidth allocation game was the last thing (that we will consider) in terms of evaluating quality of Nash equlibria. Starting next time, we will switch topics and consider algorithmic problem of finding Nash equilibria. This is an interesting problem on the boundary of Game Theory and Computer Science and as Christos Papadimitriou puts it: "...existence of polynomial time algorithm for computing a mixed Nash equilibrium *in a two person game* is famously, and quite astonishingly, open.

As there is not much to offer in algorithmic problem of finding Nash equilibria, we will consider other *reasonable* concepts of equilibria and see algorithms to find them. Towards the end of the course, we will also talk about convergence of best response dynamics which is used to prove existence of Nash equilibria.

References

[JT04] Johari, R., and Tsitsiklis, J.N. (2004). Efficiency loss in a network resource allocation game. Mathematics of Operations Research 29 (3): 407–435.