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| **CS 684 Algorithmic Game Theory** | **Scribe: Prakash Linga** |
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In today's class we will start to talk about the limits of what can be done truthfully in auctions. We will see that VCG is the only possible algorithm in many cases.

# 1 Road map

- `Arrow's impossibility theorem on voting` : Today's class
- `Gibbard-Satlerthwaite` : Impossibility of truthful mechanisms without money transfer.
- `Robert's` : In a certain general model, only truthful mechanism is weighted VCG.

# 2 Arrow's Impossibility Theorem

## 2.1 Problem

There are $N$ users (players or voters). Each player has an ordered list of preferences ($x > y > z > \cdots$) on $\geq 3$ candidates (or resources). Can we select a winner such that

1. `Unanimity preserving`: If all players have $x > y$ then $y$ does not win.
2. `Independence of irrelevant alternatives or Pair-wise preference relationship`: Given some preferences, let's say the result is $x$ and not $y$. A change in preference of $z$ will not cause $y$ to win.
3. `Non-dictatorial`: The solution will not be just copying a certain player's list.

## 2.2 A simple algorithm proposal

Define $x > y$ if majority prefers $x$ over $y$.

This is not good enough because of `Condescet paradox`. Consider three users with the following preference lists:

- User 1: $a < b < c$
- User 2: $b < c < a$
- User 3: $c < a < b$

From the above preference lists, $a < b < c < a$ and hence it is not possible to select a winner.

`Corollary`: With $\geq 3$ candidates there is no truthful election scheme.

## 2.3 Theorem

*There is no selection rule satisfying the above three properties*

We will show that any selection rule satisfying rules 1 and 2 is dictatorial.

Now, any selection rule satisfying rules 1 and 2 can be extended to generate a total order $>$ of candidates such that

- All players prefer $x$ to $y$ implies $x > y$.
- Outcome of $x > y$ depends only on users preference between $x$ and $y$.

### How to generate a total order on candidates

Define overall $x > y$ as follows: Given user $i$ preference $a > \cdots > x > \cdots > y > \cdots$, change this to put $x$ and $y$ at the begining. The modified preference list would be $x > y > a > \cdots$.

- **Claim**: $x$ or $y$ wins.

  **Proof**. Follows from axiom 1. ■

  We will define that $x > y$ overall if $x$ wins.

- **Claim**: Above definition of $>$ is an order i.e $x > y \land y > z \Rightarrow x > z$

  **Proof**. Move $x$, $y$ to the top two places. Now $x$ wins by definition. Now slide $y$ to its place among the the top 3.

  We claim that the winner is now $x$ or $z$ and not $y$. This follows from axion 2.

  If $x$ wins then $x > z$ (i.e decreasing $y$ is irrelevant) and this is what we want to prove.

  If $z$ wins then $z > y$ (i.e decreasing $x$ is irrelevant). A contradition. ■

**Definition**: Given that $x > y$, a subset $S$ is *decisive* if all $i \in S$ prefer $x > y$ and all $i \notin S$ prefer $y > x$ implies outcome is $x > y$.

Note that outcome only depends on $x$ over $y$ choices, based on axiom 2.

**Lemma**: $S$ decisive for $x > y$ then $S$ is also decisive for all $a > b$.

**Proof**. We will show that: $S$ decisive for $x > y$ implies also decisive for $x > b$.

$x, b$ preference does not depend on $y$. Therefore, choose $y > b$. Consider the following setup:

- $S : x > y > b$
- $\bar{S} : y > b > x$

$x > y$ in $S$ and $y > x$ outside $S$, so we know that $x > y$ overall (as $S$ is decisive). Also we chose $y > b$ everywhere, so $y > b$ by axiom 1. Therefore $x > b$ by transitivity. Now $x > b$ in $S$ and $b > x$ outside of $S$, so $S$ is decisive for $x > b$.

Choose $x > a$ and use a similar argument as above argument with $x > b$ to get $S$ is decisive on $a > b$ ■

**Observation**: $S$ or $\bar{S}$ is decisive for any $S$.

**Lemma**: $S$ and $S'$ decisive implies $S \cap \bar{S}$ is decisive.

**Proof**. To prove this consider 3 elements $x$, $y$, and $z$. Assume that we have the following:

- $S \setminus S'$: $x > z > y$ y
- $S \cap S'$: $x > y > z$
- $S' \setminus S$: $y > x > z$
- $S \bar{\cup} S'$: $z > y > x$

Now $S$ is decisive, so we get $x > y$. Also $S'$ is decisive, so we get $y > z$. so by transitivity, we get that $x > z$. But now notice that $x > z$ is only true on $S \cap S'$ so we conclude that $S \cap S'$ is decisive.

■

### Finally, proof of Arrow's theorem

**Proof**. (**Arrow's theorem**)

Let $v \in N$ be some voter. Then either $\{v\}$ or $N - v$ is decisive. If none of $\{v\}$ for some $v \in N$ is decisive then $\cap_v N - v$ is decisive. But $\cap_v N - v = \phi$, the empty set. This contradicts axiom 1 (unanimity preserving).

If for two different nodes $v$ and $w$ the sets $\{v\}$ and $\{w\}$ are decisive, than their intercestion is decisive, which is empty. We get the same contradiction.

So it must be the case that exactly one set $\{v\}$ is decisive, and for all other $w \neq v$ the set $N - w$ is decisive. But then all sets containing $v$ are decisive (as they are intersections of sets of the form $N - w$). This means that $v$ is the dictator.

■

Note that Arrow's impossibility theorem is a weak result because there are no weights and no bribes.