Algorithmic Game Theory	**Problem Set 2**
CS 684 Spring 2004	**Due Wednesday, March 17, 2004**

Solve at least 3 of the following 4 problems. You may solve all 4 for extra credit. The problems are of varying difficulty, but are worth equal credit. Taking scribe notes for class can replace solving one of the problems. We will maintain a FAQ for the problem set on the course Web page. You may use any fact we proved in class without proving the proof or reference.

Project proposal due: Submit a brief proposal for your final project before spring break (by Friday March 19). The project proposal is a few paragraph long explanation on what you are hoping to accomplish. The goal of asking for the proposals is to get each group started on thinking about the projects, and also to allow early feedback on the plans.

There are many different types of projects possible. You can consider a game, and propose to work on understanding the algorithmic issues associated with this game, such as how to find an equilibria, are there many equilibria, do natural plays find such equilibria, what are the quality of the worst or best equilibria, etc. I think the most interesting projects would model some phenomena in computer science as a game, and analyze a simplified version. Projects may be theoretical or experimental in nature. Alternatively, your project may be an extended survey of the literature in a sub-area. If you choose to do a survey, you should make sure that you understand the papers well, and can compare the results, and techniques of different papers, explain their advantages and disadvantages. You can work on this project in groups of at most 3.

(1) In class we proved that in a game with a finite set of players, where each player has a finite set of pure strategies, the game has a Nash equilibrium. Our algorithm was based on Brower's fixed point theorem: we showed that there is a sequence of points that converges to a Nash equilibrium. However, our proof was not algorithmic. The Nash equilibrium we found was the limit of an infinite sequence of multicolored simplices. This line of argument does not lead to a finite algorithm. It is a major outstanding open problem whether a Nash equilibrium in such a game can be found in polynomial time. In this problem, we consider the special case with two players. To be more formal, assume that there are 2 players, and player i chooses between n_i pure strategies. Assume that the game is given by listing the payoff for each player for each $n_1 \times n_2$ possible plays (this is the traditional payoff matrix). Give a finite algorithm for finding a Nash equilibrium for this case. Your algorithm may run in exponential time.

(2) Consider a 2-person game in the matrix form also used in the previous problem. Assume that both players have n pure strategies. In a Nash equilibrium a player may be required to play a mixed strategy that gives non-zero probability to all (or almost all) of his pure strategies. Strategies that mix between so many pure options are hard to play, and also hard to understand. The goal of this problem is to show that one can reach an almost perfect Nash equilibrium by playing strategies that only choose between a few of the options.

As in class, we will use p^j to be the probability distribution for player j, so p^j_i is the probability that player j will use his ith pure strategy. The support of a mixed strategy p^j for player j is the set i such that $p^j_i > 0$, i.e., the number of different pure strategies being mixed. We will be interested in solutions where each job has a strategy with small support. Unfortunately, some Nash equilibria may give players strategies with large support.

For a given $\epsilon > 0$, we will say that a set of mixed strategies p^1, p^2 is ϵ-approximate Nash if for both players $j = 1$ or 2, and all other strategies $\bar{p}^j$ for this player, the expected payoff for player j using strategy $\bar{p}^j$ is at most ϵM more than his expected payoff using strategy p^j, where M is the maximum payoff.

Show that for any fixed $\epsilon > 0$ and any 2 player game with all nonnegative payoffs, there is an ϵ-approximate Nash equilibrium such that both players play the following simple kind of mixed strategy: For each player j the strategy selects a subset $\hat{S}_j$ of at most $O(\log n)$ of player jth pure strategies, and makes player j select one of the strategies in $\hat{S}_j$ uniformly at random. The set $\hat{S}_j$ may be a multi-set, i.e., may contain the same pure strategy more than once (such a strategy is more likely to be selected by the random choice). The constant in the $O(.)$ notation may depend on the given parameter ϵ.

Hint: consider any mixed Nash strategy with possibly large support, and try to simplify the support by selecting the subsets $\hat{S}_j$ for the two players based on this Nash strategy.

(3) In class we discussed Evolutionary Stable Strategies. Here is a 2 person game with no Evolutionary Stable Strategy.

γ, γ	$1, -1$	$-1, 1$
$-1, 1$	γ, γ	$1, -1$
$1, -1$	$-1, 1$	γ, γ

In this game the players choose between 3 pure strategies. Assume $0 < \gamma < 1$. The unique Nash equilibrium is to mix (1/3, 1/3, 1/3) between the 3 pure strategies. This Nash is not evolutionary stable, and hence this game has no evolutionary stable strategy.

Consider a symmetric two person game, where both players have only two possibly pure strategies, call them A and B. The game being symmetric means that there are 4 different payoffs: the payoff for both players for playing strategy A, or B respectively, and the payoff if one plays A, and the other plays B. Assume that all four of these payoffs are different. Show that such a game always has an Evolutionary Stable Strategy.

(4) The classical Bertrand game is the following. Assume that there are n companies competing for customers by producing the same product. Each company i can determine a production level q_i, so there will be $q = \sum_i q_i$ product on the market. Now demand for this product, of course, depends on the price, and once q amount is on the market, price will settle so that all q amount is sold. Assume that we are given a "demand-price curve" $p(d)$ is the price that allows all q units to be sold. Assume that $p(d)$ is a monotone decreasing, differentiable function of d. With this definition, the income of the firm i will be $q_i p(q)$. Assume that production is very cheap, and so each firm will produce to maximize this income.

(a) Show that the total income for a monopolistic firm, can be arbitrarily higher than the total income of many different firms sharing the same market. Hint: this is true for almost all price curves, you may want to use for example $p(d) = 1 - d$.

(b) Assume that $p(d)$ is twice differentiable, monotone decreasing, and $p''(d) \leq 0$. Show that the monopoly income is at most n times the total income of the n competing companies.