CS 684 Algorithmic Game Theory	**Scribe: Ara Hayrapetyan**
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1 Market Equilibrium

Our next topic where we will spend a few lectures will be games related to market equilibria. In today's game we will have k players who bring n different kinds of goods to the market. The players obey a *pricing mechanism*, which states that given a price $p_i > 0$ for good i, the players will sell the items they brought and will purchase goods so as to maximize a personal utility.

The input to the problem is a vector $e_1, \ldots, e_k \in R^n$, where each element e_i is itself a vector of quantities that player i brought to the market. We are also given utility functions $u_i(x) : R^n \mapsto R$ denoting the utility that a player i receives from purchasing quantities of goods described by the vector x. We assume that $u_i(x) > 0$ and that $u_i(x)$ is strictly monotone and strictly concave. We will give an intuition on these assumptions shortly.

The question we would like to answer is whether there are a set of market clearing prices. Formally we desire a set of prices $p = p_1, \ldots, p_n$ for the goods such that if players sell their goods at these prices and then purchase (with the revenue made from the selling) a set of goods so as to maximize their personal utility, there will be no envy (no good will be demanded more than it is supplied) and the market will be cleared (no good will be supplied more than demanded).

Given a set of prices p player i gets $p \cdot e_i$ from selling his goods (the $\cdot$ denotes a dot product). Then the vector of goods x_i that he wants is the solution of the following system:

$$max \quad u_i(x) \tag{1}$$

$$s.t. \quad p \cdot x \leq p \cdot e_i \tag{2}$$

The set of prices p is then at an equilibrium if $\sum e_i = \sum x_i$.

A brief reason why it makes our job easier to assume strictly monotone utility functions is that the strictly monotone utility ensures that at price 0 everyone will want infinite quantities of the good, therefore the solution will produce strictly positive prices.

And the reason why we want strictly concave is that it makes the solution to the above system unique, since if x_1 and x_2 are two feasible optimal solutions then $\frac{x_1+x_2}{2}$ is a strictly better solution.

Now comes our main theorem.

Theorem 1 *Equilibrium prices do exist under the above assumptions.*

Proof: We want to prove this using the Kakutani fixed point theorem. So we have set up the set function. The overall idea is that if product j is over-produced its price should go down and if it is under-produced its price should go up. First, without loss of generality we can assume that $\sum p_i = 1$ (scale), so we have a simplex. Let z denote the excess in the market, i.e. $z = \sum x_i - \sum e_i$. Our intuition tells us that we want the price go up on the products whose component in z is positive, and down if it is negative.

Let $f(p) = \{q > 0 \mid \sum q_i = 1 \ and \ q_j > 0 \ only \ if \ j \ is \ a \ maximum \ component \ of \ z\}$ for all $p > 0$. We claim that this produces the desired fix-point.

First notice that the prices p define f in an alternate way, i.e. $f(p) = \{q \mid qz \geq q'z \, \forall q' \; with \; \sum q_i = \sum q'_i = 1\}$. Unfortunately our function f defined thus far is not closed, since we defined it on strictly positive prices. We fix it by saying that for those price vectors p with at least one component equal to 0, the function f maps them to the set of vectors whose dot product with p is 0 (notice that we did this construction so as to ensure no fix-point among such sets since $p \cdot p$ is never 0).

If we can show that f is continuous, then we would be done, since as we just observed a fix-point p^* of f must have strictly positive prices, but then $p \in f(p)$ by definition puts price on every component, and thus every component has maximum excess therefore every component has excess 0.

We are left to show the continuity of f. Recall that for f to be continuous given a converging price sequence $p^1, p^2 \ldots \to p$ and a converging sequence of elements $q^i \in f(p^i)$ where q^i converges to q, we need to show that $q \in f(p)$. There are two cases to consider here, when $p > 0$ and when p has a zero component.

First consider the case when $p > 0$. For the p^i's the set of corresponding z^i's also clearly converges to z because z is a continuous function on prices and utilities and those are all continuous. Now we use the alternate definition of f. Since q^i is in $f(p^i)$, we have $q^i \cdot z^i \geq q' \cdot z^i$ for all q'. Again because of continuity, this inequality also holds in the limit as well, therefore $q \cdot z \geq q' \cdot z$ for all q', which means that $q \in f(p)$.

Due to lack of time, we will not discuss the case when p has a zero component in detail. For this case, the original proof of $f(p)$ seems more appropriate. We need to show that for every non-zero coordinate of q, the corresponding coordinate of p is zero. The idea is that once p^i has a small enough coordinate, than the demand in this product will be very high, and as a result, the maximum coordinates of z will be from among these products, so all other coordinates of q^i must be zero.