Bandwidth Sharing

The **bandwidth sharing game** is defined as follows:
There is a single link with capacity B, k users, and a utility function $U_i(x)$ for all users i. U_i must be monotone increasing and concave. For proof purposes, we also make the additional weak assumptions that U_i is strictly concave and differentiable.

There is a pricing mechanism in this game:
Given a price p, each user solves the following problem

$X_i = \max (U_i(x) - px)$

The user is maximizing the difference between the value of the bandwidth he is getting, and the price he is paying for it. Since we assume the utility function is concave, each X_i is unique.

The price p is at equilibrium if $\sum X_i = B$.

Facts (from last lecture):

1. Equilibrium price exists
2. At equilibrium price p, the resulting X_i's maximize $\sum U_i(x_i)$. This is the social welfare optimum.
3. Nash equilibrium of the following game results in equilibrium price.

The Kelly mechanism for playing this game:

User i offers W_i money, so the resulting price is $p = \frac{\sum_i Wi}{B}$

Assumption: Users are "**price takers**" – they ignore their own effect on the price (explained by either the users being dumb, or, mathematically, if there is a large number of them, no individual user has a significant impact on the price).

So the best response in this case is $\mathbf{X_i = \max (U_i(x) - px)}$. So they offer $\mathbf{W_i = X_i * p}$.

Johari & Tsitsiklis Game:

This is the natural version of the game, in which users are "**price anticipators**" – they take the price formula into account when determining their W_i's.

User i offers W_i and gets $\frac{B * Wi}{\sum_j Wj}$ (proportional sharing).

User i happiness is $U_i * \frac{B * Wi}{\sum_j Wj}$ $-W_i$

This game is different, so it doesn't necessarily result in a social welfare optimum.

Question: How does allocation at Nash compare to the social optimum in this game?

Theorem (Johari & Tsitsiklis):

The worst ratio is ¾ (the social value at Nash is no worse than ¾ the value of the social optimum).

What makes a solution a Nash?
Denote price as:

$$p = \frac{\sum_j Wj}{B}$$

$$X_i = \frac{Wi}{p}$$

User problem: Given all W_j, $j \neq i$, maximize $U_i\left(\frac{B * Wi}{Wi + \sum_{j \neq i} Wj}\right) - W_i$

User is at optimum if the derivative is equal to 0, so if we use the chain rule to take the derivative if the above function we get:

$$Ui'\left(\frac{B*Wi}{Wi+\sum_{j\neq i}Wj}\right)\left(\frac{B*Wi}{Wi+\sum_{j\neq i}Wj}\right)'-1=0$$

We note that $\frac{B*Wi}{Wi+\sum_{j\neq i}Wj}=\frac{Wi}{p}=Xi$. Substituting this and taking the derivative of the second term we get

$$Ui'(Xi)*\left(\frac{1}{p}-\frac{B*Wi}{\left(Wi+\sum_{j\neq i}Wj\right)^2}\right)=1$$

Now multiply by the price p on both sides to get:

$$Ui'(Xi)'*(1-\frac{Xi}{B})=p$$

What is the equilibrium point?

In summary, a Nash allocation is values X_1, X_2 …. X_k and a prize p so that the allocations sum to B, and they satisfy the above equations.

Corollary: A (deterministic) Nash exists.

Since we assumes that Ui' monotone and continuous, the function $Ui'(Xi)'*(1-\frac{Xi}{B})$ is also continuous, and so for any prize p, we get an allocation Xi that satisfies the equation. When p is low the sum of the allocations is less than B. raising p continuously raises the sum, so there is a value at which the sum equals B.

Note that if someone has a small share of the bandwidth, their change in contribution won't affect the price much, but if they have a large share already, additional contribution may depress the price by a lot.

How good is the equilibrium point?

It will come out in the proof of the theorem as follows:

Let the Nash allocation be X_1, X_2 …. X_k

Some people get more than in Opt, and some people get less (since the sum in both is B).

Fact 1: Worst case utility is linear.

In determining Nash and its value, $U_i'(X_i)$ and $U_i(X_i)$ are the only things that matter. So if we create an alternate utility function Z(x) with:

$$\mathbf{Zi(x) = U_i (X_i) + (x-X_i)*U_i'(X_i),}$$

then we can:
Claim: Utilities Zi have same Nash Xi, and the same value at Nash. The Opt value only improves because Zi(x)>=Ui(x) for all x, as the function Ui is concave by assumption

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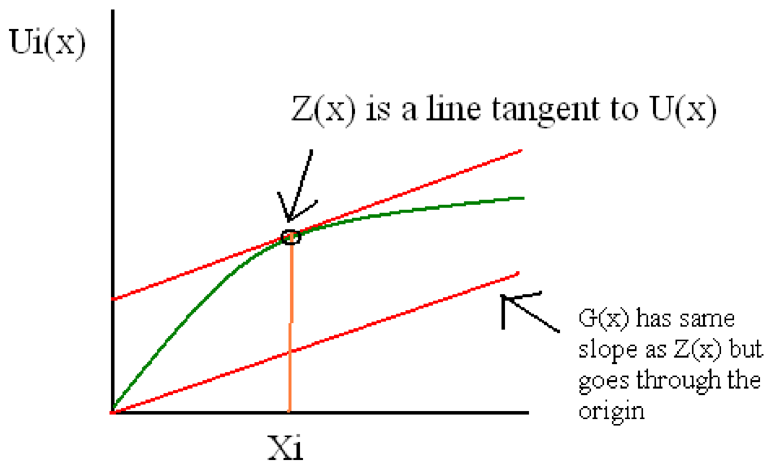
Fact 2: Worst case is when Gi(x) = a_i*x (push the line down to go through the origin)



Proof:
Let a new utility function Gi(x) = Ui'(Xi)*x
The Nash is the same place, Xi as before (same reasoning as for fact 1, the only thing that is needed to prove that Xi is a Nash is the value Gi'(Xi) which is the same as Ui'(Xi))

By the same argument the location of Opt in G is the same as the location of opt in Z, as it only depends on the derivative, and pushing the line down doesn't change its slope, so Opt is still in the same location.

What about its values though? A constant gets subtracted from both the Nash and the Opt value. Denote the constant as $\mathbf{C = \sum_i Zi(0).}$ It's subtracted from both Nash and Opt.
The old ratio of Nash to Opt is: $\frac{N}{O}$ and the new ratio is $\frac{N-C}{O-C}$
The new ratio is worse than the original ratio.

If all utilities are linear, the social optimum involved giving the user with the steepest curve all of the bandwidth, and giving nothing to all the other users. So lets say user 1 is the user with the steepest curve, which is:

U_1'(x) = $\dfrac{p}{1-\dfrac{X_1}{B}}$

In Nash, we give some amount to other users. The ratio get worse if we make all other user's curves less steep. All other users have Ui'≥p, so the worst case is what all others have **Uj'(x) ≈ p** (and all have very small amounts of bandwidth).

So Opt = $\left(\dfrac{p}{1-\dfrac{X_1}{B}}\right) * B$

Nash = $\left(\dfrac{p}{1-\dfrac{X_1}{B}}\right) * X_1 + (B - X_1) * p$

Hence the ratio Nash/Opt is:

$$\left(\frac{B}{X_1} + \frac{\dfrac{B-X_1}{B}}{1-\dfrac{X_1}{B}}\right) = \left(\frac{\dfrac{1}{(1-a)}}{\dfrac{a}{1-a} + (1-a)}\right) \qquad \text{(by letting a} = \frac{X_1}{B}\text{)}$$

Simplifying we get:

Nash/Opt = $\dfrac{1}{a+(1-a)^2}$

Hence Opt/Nash = $a+(1-a)^2$

What a will cause this ratio to have the worst possible value? Taking the derivative and setting it to 0 we see that 1 – (2*(1-a)) = 0, so a = ½. Plugging that value into the ratio equation, we see the worst ratio is exactly ½ + ¼ = ¾