CS 684 Algorithmic Game Theory	Scribe: Sumit Kunnumkal
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1 Correlated equilibria

In this lecture we will look at correlated equilibria. We start with an example. Consider the game *Battle of Sexes*. Recall, that this game has two players who want to go either to a baseball game (B) or a softball game (S). The payoff matrix is as given below. The first entry in a cell denotes the player I's reward and the second entry denotes player II's reward. We can see that (S,S) and (B,B) constitute deterministic Nashes, with rewards (2,1) and (1,2), respectively. A randomized Nash consists of player I choosing S with probability 2/3 and B with probability 1/3 and player II choosing S with probability 1/3 and B with probability 2/3. The expected reward in this case is (2/3, 2/3), which is fair but lower than the worst outcomes in the deterministic Nashes.

	S	B
S	2,1	0,0
B	0,0	1,2

Now, consider a situation where a "trusted" authority flips a fair coin and based on the outcome of the coin toss, tells the players what they should do. So, for example, if the coin shows heads, player I is told to choose baseball and player II is told to choose baseball. Similarly, both players are told to choose softball when the outcome is tails. **(??)** It is important to note that no individual party has an incentive to deviate from what they are told to do. In this case, when player I is told to choose B, he knows that player II is told to choose B as well. So, player I has no incentive to deviate and switch to S as the payoff would be lower (0 compared to 1). The advantage of following such a procedure is that the expected rewards are now higher – (3/2, 3/2) compared to that of (2/3, 2/3) from the mixed NE.

Outcome	Probability
(B,B)	1/2
(S,S)	1/2

A more interesting example of correlated equilibria is the game of *Chicken*. This is again a two player game with the payoff matrix as shown below. In this case, the worst outcome occurs when both players dare (D,D). The deterministic Nashes are (D,C) and (C,D) with rewards (7,2) and (2,7), respectively. **(??)** A randomized Nash has players I and II choosing C and D with probabilities 2/3 and 1/3, respectively. This randomized Nash has an expected reward of $4\frac{2}{3}$ for each player.

	C	D
C	6,6	2,7
D	7,2	0,0

Now, let's look what happens in the case of a correlated equilibrium. As before, a trusted party tells each player what to do based on the outcome of the following experiment:

Outcome	Probability
(C,D)	1/3
(D,C)	1/3
(D,D)	1/3

We note once again that the trusted party only tells each player what he/she is supposed to do. The trusted party does not reveal what the other player is supposed to do. It is a correlated equilibrium if no player wants to deviate from the trusted party's instruction.

(??) So, in the *Chicken* example, if the trusted party tells player II to dare, then II has no incentive to deviate. This is because II knows that the outcome must have been (C,D) and that player I will obey the instruction to chicken. Next, let us consider the case when player II is told to chicken. Then player II knows that the outcome must have been either (D,C) or (C,C), each happening with equal probability. II's expected payoff on playing C conditioned on the fact that II is told to chicken is $\frac{1}{2} * 6 + \frac{1}{2} * 2 = 4$. In the above expression, 6 is the payoff from I also playing C, i.e., the outcome was (C,C) and 2 is the payoff II gets when I plays D, i.e., outcome was (D,C). If player II decides to deviate, i.e., play D when told to play C, then the expected payoff is $\frac{1}{2} * 7 + \frac{1}{2} * 0 = 3.5 < 4$. So, the expected payoff on deviating is lower than the payoff on obeying the instruction of the trusted party. Therefore player II doesn't deviate. Since the game is symmetric, player I also has no incentive to deviate from the instruction of the trusted party. Note that in the case of the correlated equilibrium, the expected reward for each player is $= \frac{1}{3} * 7 + \frac{2}{3} * 4 = 5$. This is higher than the expected reward of $4\frac{2}{3}$ in the randomized Nash. Therefore, rewards can be made better by correlation.

(??) Finally , we consider the general case of k player matrix games, where player i has n_i pure strategies. Then, we can find a correlated equilibrium in time polynomial in $n_1 n_2 \ldots n_k$ using linear programming.

Remarks

- We consider only atomic games, so that the number of strategies is finite.
- We contrast this with the problem of finding a Nash equilibrium for a general game, for which no polynomial time algorithm is known.

(??) For a correlated equilibrium, we need to find a probability distribution on the set of all possible strategies. Let s_i be an element of the set of pure strategies of player i and let $s = (s_1, \ldots, s_k)$. Also let $\text{payoff}_i(s)$ be the payoff to player i when strategy s is followed by the players. $p(s_1, \ldots, s_k)$ denotes the probability with which the trusted party observes the event $(s_1, \ldots, s_k)$, in which case players $1, \ldots, k$ are told to play strategies $s_1, \ldots, s_k$, respectively. To ensure that a correlated equilibrium results, no player should have an incentive to deviate from the instruction. So, if player i is told to play $\overline{s_i}$, then then all other strategies for that player should have no better outcome. Thus we want the expected payoff of strategy $\overline{s_i}$ to be at least as great as the expected payoff when player i alone switches to some other strategy $s_i^{'}$.

$$\sum_{s:s_i=\overline{s_i}} p(s)\text{payoff}_i(s) \geq \sum_{s:s_i=s_i^{'}} p(s)\text{payoff}_i(s_1, .., s_i^{'}, ..s_k)\ \forall s_i^{'} \tag{1}$$

In addition, since $p(s)$ is a probability distribution, $p(s) \geq 0$ and $\sum p(s) = 1$. The above inequalities define a linear program which can be solved to get a correlated equilibrium. We note:

- The number of variables is $n_1 n_2 \ldots n_k$.
- We have to divide both sides of (1) by $\sum_{s:s_i=\overline{s_i}} p(s)$ so that it represents the expected payoffs.

- If we want to find the "best" correlated equilibrium, then we can simply introduce an appropriate objective function in the LP. For example, if the objective function is social welfare, the objective would be to maximize $\sum_s p(s) \sum_i \text{payoff}_i(s)$

We conclude with the following facts:

Fact 1 *All mixed Nashes are correlated, so correlated equilibria exist.*

Fact 2 *All convex combinations of mixed Nashes are also correlated.*