CS 684 Algorithmic Game Theory	Scribe: David Martin
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Non-atomic Routing Games

Nash Cost vs. Optimal Cost

Last time, we started analyzing worst-case Nash/Opt cost ratios for various delay functions for non-atomic routing games.

Today, we will look at two types of delay functions that can lead to arbitrarily high the Nash/Opt cost ratios.

However, we will then show that, for a modified the optimality condition, the Nash/Opt cost ratio is indeed bounded regardless of the delay function.

Polynomial Delay Functions (x^n)

Setup

Consider graph $G = (V, E)$, where $V = \{s_1, t_1\}$ and $E = \{e_1, e_2\}$ and e_1 and e_2 are distinct edges from s_1 to t_1.

The delay functions are:

$$\begin{aligned} d_{e_1}(x) &= 1 \\ d_{e_2}(x) &= x^n \end{aligned}$$

The cost function C is:

$$\begin{aligned} C(f) &= (1-x).1 + x.x^n \\ &= 1 - x + x^{n+1} \end{aligned} \quad \text{, where } x = f(e_2)$$

Nash Cost

Nash flow sends all flow on edge e_2, and hence Nash delay is 1.

Optimal Cost

Let x be the flow on edge e_2 for the optimal flow.

$$\begin{aligned} 0 &= C'(f^*) \\ &= ((1-x).1 + x.x^n)' \\ &= (n+1).x^n - 1 \end{aligned}$$

Thus $x = (n+1)^{-1/n}$.

Thus optimal cost is:

$$\begin{aligned} C(f^*) &= 1 - x + x^{n+1} \\ &= 1 - x.(1 - x^n) \\ &= 1 - (n+1)^{-1/n}.n/(n+1) \\ &= 1 - n/(n+1)^{1+1/n} \end{aligned}$$

Nash/Opt Ratio

Notice that as n increases, $1 - n/(n+1)^{1+1/n}$ goes to 0.

Hence the Nash/Opt cost ratio increases unboundedly as n increases.

Queuing Delay Functions **($1/(u-x)$, where u is the capacity)**

Setup

Consider graph $G = (V, E)$, where $V = \{s_1, t_1\}$ and $E = \{e_1, e_2\}$ and e_1 and e_2 are distinct edges from s_1 to t_1.

The delay functions are:

$$\begin{array}{rcl} d_{e_1}(x) & = & 1/\varepsilon \\ d_{e_2}(x) & = & 1/(1+\varepsilon-x) \end{array}, \text{where } \varepsilon > 0$$

Remark. The ε prevents Nash equilibrium from requiring an infinite amount of flow.

The cost function C is:

$$\begin{array}{rcl} C(f) & = & (1-x).1/\varepsilon + x.1/(1+\varepsilon-x) \\ & = & (1-x)/\varepsilon + x/(1+\varepsilon-x) \end{array}, \text{where } x = f(e_2)$$

Nash Cost

Nash flow sends all flow on edge e_2, and hence Nash delay is $1/\varepsilon$.

Optimal Cost

Let x be the flow on edge e_2 for the optimal flow.

$$\begin{array}{rcl} 0 & = & C'(f^*) \\ & = & ((1-x)/\varepsilon + x/(1+\varepsilon-x))' \\ & = & -1/\varepsilon + (1+\varepsilon)/(1+\varepsilon-x)^2 \end{array}$$

Thus $x = 1 + \varepsilon - (\varepsilon.(1+\varepsilon))^{1/2}$.

Thus optimal cost is:

$$\begin{array}{rcl} C(f^*) & = & (1-x)/\varepsilon + x/(1+\varepsilon-x) \\ & = & (\varepsilon - (\varepsilon.(1+\varepsilon))^{1/2})/\varepsilon + (1+\varepsilon-(\varepsilon.(1+\varepsilon))^{1/2})/(\varepsilon.(1+\varepsilon))^{1/2} \\ & = & 2.((1+1/\varepsilon)^{1/2} - 1) \end{array}$$

Nash/Opt Ratio

The Nash/Opt ratio is $(1/\varepsilon)/[2.((1+1/\varepsilon)^{1/2}-1)] = 1/[2.((\varepsilon^2+\varepsilon)^{1/2}-\varepsilon)]$.

Hence the Nash/Opt cost ratio increases unboundedly as ε goes to 0.

A Modified Optimality Condition

We have thus seen two cases (Polynomial Delay & Queuing Delay) where the worst Nash/Opt cost ratio can be arbitrarily high.

We will now consider a type of optimality for which the Nash/Opt cost ratio is always bounded by 1.

Theorem (Roughgarden-Tardos). *Let f^N be a Nash flow. Let f^* be an optimal flow for which each user sends 2 units of flow instead of 1. Then:*

$$C(f^N) \le C(f^*)$$

Proof.

We define an artificial delay function $\widehat{d_e}$ as follows:

$$\widehat{d_e}(x) = \begin{cases} d_e(f^N(e)) & \text{if } x \leq f^N(e) \\ d_e(x) & \text{otherwise} \end{cases}$$

Notice that $\widehat{d_e}$ is continuous, non-negative, and increasing (since d_e was too).

Let $\widehat{C}$ be the cost function associated with this new delay function. That is,

$$\widehat{C}(f) = \sum_{i=1}^{k} \widehat{d_e}(f(e)).f(e)$$

We will now prove two subclaims that will help us to bound $\widehat{C}(f^*)$ from above and below in terms of $C(f^*)$ and $C(f^N)$.

Subclaim 1. *If f is a flow that sends 2 units from s_i to t_i (for each i), then:*

$$\widehat{C}(f) \geq 2.C(f^N)$$

Proof.

Recall that, since f^N is a Nash flow, for any $i = 1, \cdots, k$, the d_e-delay over any s_i-t_i path is the same (say D_i). So we have:

$$\begin{array}{rcll} C(f^N) & = & \sum_{e \in E} f^N(e).d_e(f^N(e)) & \\ & = & \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_p^N.d_P(f^N) & \text{(we proved this in a previous lecture)} \\ & = & \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_p^N.D_i & \text{(since } f^N \text{ is a Nash flow)} \\ & = & \sum_{i=1}^{k} D_i. \sum_{P \in \mathcal{P}_i} f_p^N & \\ & = & \sum_{i=1}^{k} D_i.1 & \text{(since } f^N \text{ sends 1 unit of flow from } s_i \text{ to } t_i) \\ & = & \sum_{i=1}^{k} D_i & \end{array}$$

Consider any flow f that sends 2 units from s_i to t_i for each $i = 1, \cdots, k$.

Now, for any edge e, by definition, $\widehat{d_e}$ is bounded from below by $d_e(f^N(e)))$.

Thus, for any s_i-t_i path P, $\widehat{d_P}$ is bounded from below by $d_P(f^N) = D_i$. So we have:

$$\begin{array}{rcll} \widehat{C}(f) & = & \sum_{e \in E} f(e).\widehat{d_e}(f(e)) & \\ & = & \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_p.\widehat{d_P}(f) & \text{(we proved this in a previous lecture)} \\ & \geq & \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_p.D_i & \text{(since } \widehat{d_P} \text{ is bounded below by } d_P(f^N) = D_i) \\ & = & \sum_{i=1}^{k} D_i. \sum_{P \in \mathcal{P}_i} f_p & \\ & = & \sum_{i=1}^{k} D_i.2 & \text{(since } f \text{ sends 2 units of flow from } s_i \text{ to } t_i) \\ & = & 2. \sum_{i=1}^{k} D_i & \end{array}$$

Thus $\widehat{C}(f) \geq 2.C(f^N)$. ■ **(Subclaim 1)**

Subclaim 2. *If f is any flow, then:*

$$\widehat{C}(f) \leq C(f) + C(f^N)$$

Proof.

Consider any flow f.

Consider any edge $e \in E$.

Case. $f(e) > f^N(e)$.

$$\begin{array}{lll} & [\widehat{d}_e(f(e)) - d_e(f(e))].f(e) & \\ = & [d_e(f(e)) - d_e(f(e))].f(e) & \text{(by definition of } \widehat{d}_e) \\ = & 0 & \\ \leq & d_e(f^N(e)).f^N(e) & \text{(since } d_e \text{ non-negative)} \end{array}$$

Case. $f(e) \leq f^N(e)$.

$$\begin{array}{lll} & [\widehat{d}_e(f(e)) - d_e(f(e))].f(e) & \\ = & [d_e(f^N(e)) - d_e(f(e))].f(e) & \text{(by definition of } \widehat{d}_e) \\ \leq & [d_e(f^N(e)) - d_e(f(e))].f^N(e) & \text{(since } f(e) \leq f^N(e) \text{ in this case)} \\ \leq & d_e(f^N(e)).f^N(e) & \text{(since } d_e \text{ non-negative)} \end{array}$$

In both cases we see that for every edge $e \in E$, we have:

$$[\widehat{d}_e(f(e)) - d_e(f(e))].f(e) \ \leq \ d_e(f^N(e)).f^N(e) \qquad \textbf{(2.1)}$$

Now consider the difference of the cost functions for flow f:

$$\begin{array}{lll} & \widehat{C}(f) - C(f) & \\ = & \sum_{e \in E} \widehat{d}_e(f(e)).f(e) - \sum_{e \in E} d_e(f(e)).f(e) & \text{(by definitions of } C \text{ and } \widehat{C}) \\ = & \sum_{e \in E} [\widehat{d}_e(f(e)) - d_e(f(e))].f(e) & \\ \leq & \sum_{e \in E} d_e(f^N(e)).f^N(e) & \text{(by } \textbf{(2.1)}) \\ = & C(f^N) & \text{(by definition of } C) \end{array}$$

Thus $\widehat{C}(f) \leq C(f) + C(f^N)$. ■ **(Subclaim 2)**

Proof of Theorem (contd).

From Subclaims 1 & 2, we get:

$$2.C(f^N) \leq \widehat{C}(f^*) \leq C(f^*) + C(f^N)$$

Thus $C(f^N) \leq C(f^*)$. ■ **(Theorem)**

Corollary. *Suppose that all the delays are queuing delay functions. Let f^N be a Nash flow. Let f^* be an optimal flow for twice the original capacity. Then:*

$$C(f^N) \leq C(f^*)$$