CS 684 Algorithmic Game Theory	**Scribe: Alice Cheng**
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1 Serial Cost Sharing

Recall from the previous lecture the serial cost-sharing setup: We have players, 1, ..., k, who decide on quantities $q_i \geq 0$. There is an associated cost function C, and total cost $C(q)$, where $q = \sum_{i=1}^{k} q_i$. We will assume that C is strictly convex, differentiable and monotone increasing. For each player, we have cost shares ς_i such that $\sum_{i=1}^{k} \varsigma_i = C(q)$. We also have player utilities, u_i. In the last lecture, we used a more general form for the utility functions: For each player i, we have $u_i(q_i, \varsigma_i)$ where u_i is strictly concave, differentiable, monotone increasing with respect to q_i and monotone decreasing with respect to ς_i.

In the previous lecture, we defined serial cost-sharing to be the following: Given shares $q_1 \leq q_2 \leq \ldots \leq q_k$, let player i pay cost $\varsigma_i = \frac{1}{k-i+1}[C(q_1 + \ldots + q_{i-1} + (k-i+1)q_i) - \sum_{j=1}^{i-1} \varsigma_j]$.

With the above assumptions, we showed last lecture that the serial cost-sharing mechanism yields a deterministic Nash equilibrium that is unique, and easy to derive.

2 Quality of cost sharing & Pareto Optimality

As in previous lectures, once we have a Nash solution, it is reasonable to consider the quality of the Nash. It turns out that even under a very loose definition of optimality, the serial-cost sharing solution fails to be optimal. However, this isn't a particular property of serial cost-sharing, as it turns out that no matter the cost sharing mechanism one uses, it will have a Nash that is not optimal. The goal of the remaining part of the lecture is to define a notion of quality, Pareto optimality, and to prove the following theorem, by Moulin & Shenker:

Theorem 1 *(Moulin,Shenker) (Sketch) All cost sharing mechanisms result in utilities that are "not so good".*

To make the notion of goodness more rigorous, we have the definition:

Definition 1 *A solution,* $(q_1, \ldots, q_k), (\varsigma_1, \ldots, \varsigma_k) \geq 0$ *such that* $\sum_{i=1}^{k} \varsigma_i = C(q)$, $q = \sum_{i=1}^{k} q_i$ *is* Pareto-optimal *if there is no other solution* $(q'_1, \ldots, q'_k), (\varsigma'_1, \ldots, \varsigma'_k)$ *such that* $u_i(q_i, \varsigma_i) \leq u_i(q'_i, \varsigma'_i)$ *for all users* i *and there is some user* j *such that* $u_j(q_j, \varsigma_j) < u_j(q'_j, \varsigma'_j)$.

Note that this definition only considers orders of utilities, where it only matters if one is greater than the other, but doesn't distinguish between an outcome with a slightly higher utility versus an outcome with a much higher utility.

Theorem 2 *(Moulin,Shenker) (2^{nd} version) There is no cost-sharing mechanism that yields a Pareto-optimal outcome.*

To actually prove this theorem, we will apply some simplifying assumptions. First, we will relax the strict concavity condition on the user utilities, and say that the utility of user i is $u_i(q_i) - \varsigma_i$, where u_i is a strictly concave, differentiable, and monotone increasing function on the quantity. Furthermore, we will assume that ς_i as a differentiable function of the quantities, $(q_1, \ldots, q_k)$. We will prove that no differentiable cost-sharing $\varsigma_1, \ldots, \varsigma_k$, yields a Pareto-optimal outcome.

Theorem 3 *(Moulin,Shenker) (Final version) Given $C(q)$ strictly convex, differentiable, and monotone increasing, every differentiable cost sharing $\varsigma_i(q_1, \ldots, q_k)$ (for all $i = 1, \ldots, k$, such that $\sum_{i=1}^{k} \varsigma_i(q_1, \ldots, q_k) = C(q) = C(\sum_{i=1}^{k} q_i)$) has a Nash that is not Pareto-optimal.*

To prove this theorem, we need the following lemma:

Lemma 4 *For any cost shares ς_i, cost $C(q)$ satisfying all of the same assumptions as in Theorem 3, for any quantities q_i, there exist utility functions u_i that make $(q_1, \ldots, q_k)$ Nash.*

Proof of Lemma: By definition, quantities $(q_1, \ldots, q_k)$ is Nash if given for each player i, and fixing all q_j for $j \neq i$, q_i is the quantity that maximizes $u_i(q_i) - \varsigma_i(q_1, \ldots, q_i, \ldots, q_k)$. Therefore, at the Nash equilibrium, $(q_1, \ldots, q_k)$, $u_i'(q_i) - \frac{\partial}{\partial q_i}(\varsigma_i(q_1, \ldots, q_k)) = 0$. Therefore, for $(q_1, \ldots, q_k)$ to be Nash, it suffices to have $u_i'(q_i) = \frac{\partial \varsigma_i(q_1, \ldots, q_k)}{\partial q_i}$. The left hand side of that equation is just some number, since q_i's and ς_i's are known, so we just need to find u_i with fixed slope at a single point. Clearly, such u_i will always exist. $\square$

Proof of Theorem 3: From the proof of the lemma, we know that for any Nash solution $(q_1, \ldots, q_k)$, each q_i has the property that $u_i'(q_i) = \frac{\partial}{\partial q_i}[\varsigma_i(q_1, \ldots, q_k)]$.

Consider the situation where we fix everyone's quantity and cost share except for user i. Make i increase (or decrease) his quantity q_i by some small amount. Have user i's cost share absorb the entire resulting change in the total cost. If the cost share $(\varsigma_1, \ldots, \varsigma_k)$ is Pareto-optimal, then that change cannot lead to improvement for user i. For sufficiently small increase in quantity, user i's change in utility is $u_i'(q_i)$. His new cost share is $\bar{\varsigma}_i = C(q) - \sum_{j \neq i} \varsigma_j$ (since he absorbs all of the change in cost), so his change in payment is $C'(q)$. Pareto-optimality means that $u_i(q_i) - C(q)$ is maximized. Therefore, $(q_1, \ldots, q_k)$ is Pareto-optimal $\Rightarrow u_i'(q_i) = C'(q)$ for all i.

Suppose $(\varsigma_1, \ldots, \varsigma_k)$ yields a Pareto-optimal Nash. Then, we have $u_i'(q_i) = \frac{\partial}{\partial q_i}\varsigma_i(q_1, \ldots, q_k)$ and $u_i'(q_i) = C'(q)$. Therefore,

$$\frac{\partial}{\partial q_i}\varsigma_i(q_1, \ldots, q_k) = C'(q)$$

This is independent of the utilities, so, by the lemma, we can vary user i's quantity over $0 \to q_i$ in the expression, and integrate:

$$\int_0^{q_i} C'(q_1 + \ldots + q_{i-1} + r + q_{i+1} + \ldots + q_k)dr = \int_0^{q_i} \frac{\partial}{\partial r}\varsigma_i(q_1, \ldots, q_{i-1}, r, q_{i+1}, \ldots, q_k)dr$$

By the Fundamental Theorem of Calculus, and the fact that if $q_i = 0$, $\varsigma_i(q_1, \ldots, q_i, \ldots, q_k) = 0$, this gives us

$$C(q) - C(q - q_i) = \varsigma_i(q_1, \ldots, q_i, \ldots, q_k)$$

Since the ς_i's are cost shares, $C(q) = \sum_{i=1}^{k} \varsigma_i(q_1, \ldots, q_k)$. Let $q_i = \frac{q}{k}$. By the lemma, this gives a Nash equilibrium. Then for each i, $C(q) - C(\frac{k-1}{k}q) = \varsigma_i(\frac{q}{k}, \ldots, \frac{q}{k})$. Summing over i, we get:

$$\begin{aligned} kC(q) - kC(\frac{k-1}{k}q) &= C(q) \\ (k-1)C(q) &= kC(\frac{k-1}{k}q) \\ \frac{k-1}{k}C(q) &= C(\frac{k-1}{k}q) \end{aligned}$$

This implies that C is linear, which means it violates our condition of strict concavity $\Rightarrow$ $(\frac{q}{k}, \ldots, \frac{q}{k})$ cannot be Pareto-optimal. $\square$

In particular, since the serial cost-sharing mechanism yields a unique Nash, that Nash must not be Pareto-optimal.

3 Reference

Most of the content of this lecture was drawn from:

Herve Moulin & Scott Shenker, *Serial Cost Sharing*, Econometrica, vol. 60, no. 5, pp. 1009-1037, 1992.