

Cost sharing — Shapley Value

Last time we looked at sharing value in such a way, that no subgroup of players would get more value if they would just be by themselves. Value (or cost) sharing with this property is said to be **in the core**.

The main problem with the core as we remarked last time, is that it is empty in almost all interesting cases. Today we will introduce another way of sharing cost, called the **Shapley value**, and relate this concept to the core.

1 Setup

Let N be a set of people. Let $c : 2^N \rightarrow \mathbb{R}^{\geq 0}$ be a (set)function defining the cost for each subset of N . c is assumed to be monotone (i.e. $c(S) \geq c(S')$ for all $S' \subseteq S \subseteq N$), and $c(\emptyset) = 0$.

The **goal** is to find nonnegative **cost shares** $\{x_i\}_{i \in N}$ such that

$$\sum_{i \in N} x_i = c(N) \quad (\text{budget balance})$$

2 Last lecture

Last time, our additional restrictions were

$$\sum_{i \in S} x_i \leq c(S) \quad \text{for all } S \subseteq N.$$

$\{x_i\}_{i \in N}$ satisfying this is **in the core**.

3 Axiomatic deduction of the Shapley Value

Definition The *marginal cost* for i with respect to $S \subset N$, $i \notin S$, is defined as

$$\Delta_i(S) = c(S \cup \{i\}) - c(S).$$

Consider the following axioms, that we (may) want our cost shares to satisfy.

Axiom 1 (“Dummy axiom”) If i is such that $\Delta_i(S) = c(\{i\})$ for all $S \not\ni i$, then

$$x_i = c(\{i\}).$$

Axiom 2 (“Symmetry”) If $i \neq j$ such that $\Delta_i(S) = \Delta_j(S)$ for all $S \not\ni i, j$, then

$$x_i = x_j.$$

Axiom 3 (“Linearity”) Suppose $c(S) = c_1(S) + c_2(S)$ where c_1 and c_2 are also nonnegative monotone setfunctions with $c_1(\emptyset) = c_2(\emptyset) = 0$, and $\{x_i^1\}_i$ are cost shares for c_1 -cost and $\{x_i^2\}_i$ are cost shares for c_2 -cost, then

$$x_i = x_i^1 + x_i^2$$

for all i , defines the cost shares for the c -cost.

4 Discussion of the axioms

Axiom 1 says something like if the cost of i is orthogonal (independent) of any other costs, then the cost share of i should be exactly this cost. This seems intuitively reasonable, and is actually implied in the core inequalities (namely for $S = \{i\}$ and $S = N \setminus \{i\}$, combined with budget balance).

Axiom 2 enforces some kind of fairness: if two players are (truly and utterly) indistinguishable, they should be charged the same. Note the “truly and utterly”: there are actually stronger fairness results, which we will do in class Friday.

Finally, the meaning of axiom 3 is kind of unclear, especially in the case of value, instead of costs. In the case of costs, we can think of orthogonal services (e.g. internet connection and cable television) for both of which the players have to pay to the same company. But still it is unclear (or even questionable) that we would want our cost shares to satisfy this property. It does have one big advantage: it gives rise to the following theorem.

5 Shapley Value

Theorem 1 *There is a unique way to satisfy axioms 1, 2 and 3 (and budget balance), called the Shapley Value.*

In defining the Shapley Value, we will use the notion of **ordered marginal costs**.

Definition Given an “arrival order” (permutation) σ of the elements in N , let

$$\begin{aligned} x_1^\sigma &= c(\{\sigma_1\}) \\ x_i^\sigma &= c(\{\sigma_1, \sigma_2, \dots, \sigma_i\}) - c(\{\sigma_1, \sigma_2, \dots, \sigma_{i-1}\}) \\ &= c(\{\sigma_1, \sigma_2, \dots, \sigma_i\}) - \sum_{\ell=1}^{i-1} x_\ell^\sigma \end{aligned}$$

for all i . $\{x_i^\sigma\}_i$ is said to be the **ordered marginal costs**.

Properties of ordered marginal costs:

- $x_i^\sigma \geq 0$ (since c increasing)
- $\sum_i x_i^\sigma = c(N)$ (by definition of ordering marginal costs)
- ordered marginal costs satisfy axioms 1 and 3 (provided that c_1 and c_2 use the same order)
- ordered marginal costs are not fair (think of buy-at-bulk: the players that arrive early have to pay huge setup costs, or delay on a link: the players that arrive late will have long delays).

Definition The **Shapley Value** is defined as

$$\mathbb{E}(x_i^\sigma)$$

where the expectation is taken over σ , the probability distribution of which is defined to be the uniform probability distribution of all possible permutations of N .

The following claim follows from the properties of ordered marginal costs, and the fact that σ is chosen uniformly at random.

Claim The Shapley Value satisfies axioms 1, 2 and 3.

This is half of our theorem. The other half is our next claim:

Claim No other cost sharing satisfies axioms 1, 2 and 3.

Proof Consider the following cost functions:

$$c_T(S) = \begin{cases} 0 & \text{if } S \supseteq T \\ 1 & \text{otherwise.} \end{cases}$$

The only way to satisfy axioms 1 and 2 is

$$x_i^T = \begin{cases} 0 & \text{if } i \notin T \quad (\text{by axiom 1}) \\ 1/|T| & \text{if } i \in T \quad (\text{by axiom 2}). \end{cases}$$

The next claim will finish the proof:

Claim All monotone costs c can be written uniquely as $\sum_T \lambda_T c_T$.

Proof Note that c is a $2^{|N|} - 1$ dimensional vector, and we have exactly that many c_T functions. So our claim boils down to showing that the c_T 's are linearly independent.

Suppose, by contradiction, that the c_T 's are not linearly independent, i.e. there exist nontrivial $\{\lambda_T\}_{\emptyset \neq T \subseteq N}$ such that $\sum_T \lambda_T c_T = 0$. Let S be a smallest set such that $\lambda_S \neq 0$. Note that

$$\sum_T \lambda_T c_T(S) = \lambda_S, \text{ since}$$

$$\begin{aligned} c_T(S) &= 0 && \text{when } S \not\supseteq T, \text{ and} \\ \lambda_T &= 0 && \text{when } S \supseteq T, \end{aligned}$$

contradicting the fact that every element of $\sum_T \lambda_T c_T$ equals 0. ■

6 Last remarks

First of all, the Shapley Value is in some sense orthogonal to the core. The core maybe empty, the Shapley Value always uniquely exists. In the case of a nonempty core, the Shapley Value may actually lie outside of the core altogether.

Note that the Shapley Value may be unfair: think, for example, about the case when a link is shared between two jobs, one of which is small, the other big. Since the arrival order is uniformly at random, the small job is punished for the big job half of the time, even though it has nothing to do with this big job.

Next lecture we will introduce other (stronger) fairness assumptions.