CS 684 Algorithmic Game Theory	Scribe: Christina Tavoularis
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1 Brower Fixed Point Theorem

The Brower fixed point theorem is the main tool for showing the existence of many problems in game theory for which polynomial time algorithms have not been found, and hence is an important result. The theorem can be proved using Sperner's Coloring Lemma for simplices of arbitrary dimension. In today's lecture, we will state the theorem, then prove a strong version of the coloring lemma before proving the theorem.

We will begin with a statement of the theorem. Consider the area in a k-dimensional simplex, with $k+1$ vertices such that $S_k = \{x \doteq \sum \lambda_i e_i : \lambda \geq 0 \text{ and } \sum \lambda_i = 1\} \subseteq \Re^k$, where $e_i = (0, ..., 0, 1, 0, ..., 0)$ is the unit vector in the i^{th} direction.

Theorem 1 (Brower Fixed Point Theorem) *If the function $f : S_k \to S_k$ is continuous, then there is $x \in S_k$ such that $f(x) = x$.*

For simplicity and illustrative purposes, we will consider a two-dimensional simplex ($k = 2$). Subdivide the simplex in every dimension and color the vertices with 3 colors, such that each corner has a different color. The vertices on the straight line between two corners can only be colored by the colors of the corners. A 3-colored $\triangle$ refers to a small triangle in the divided simplex such that each of its vertices is a different color.

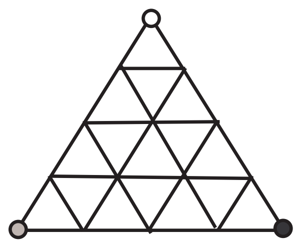

Figure 1: Subdivided 2-dimensional simplex such that each corner has a different color.

Lemma 2 (Sperner's Lemma) *3-colored $\triangle$s exist.*

In the figures, we have chosen white (w), grey (g) and black (b) to color the vertices in the divided simplex. Consider a walk on $\triangle$s that starts on the g-w side of the simplex and crosses over g-w edges only. Note that this walk will not form cycles because $\triangle$s have 3 edges and there are at most two g-w edges in a $\triangle$ (one to enter and one to exit). There are two options to terminate this walk. For option 1, the walk leads to a 3-colored $\triangle$, thus confirming existence. In option 2, the walk leaves the simplex on a g-w edge. A key observation is that the number of g-w edges on the g-w side of the simplex is odd. We would like to make use of the following Lemma which is stronger than Sperner's Lemma.

Lemma 3 *The number of 3-colored triangles is odd (this fact is needed for induction).*

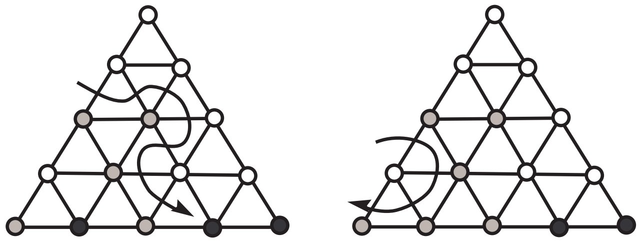

Figure 2: An example of a walk that ends at a 3-colored $\triangle$ and an example of a walk that leaves the simplex.

Proof (of Sperner's Lemma) The walk pairs up g-w edges on the side or else leads to a 3-colored $\triangle$. The fact that their are an odd number of edges on the g-w side of the simplex implies that at least one 3-colored $\triangle$ exists. ■

Proof (of lemma 3) As shown in the proof of Sperner's lemma, an odd number of walks lead to 3-colored $\triangle$s. We know that all walks lead to different 3-colored $\triangle$s, because there is exactly one g-w edge for a walk to enter from in a 3-colored $\triangle$. An important question to ask is if all 3-colored $\triangle$s are lead to by such walks? This is false! A walk terminates once it has reached a 3-colored $\triangle$. However, a reverse walk starting from a 3-colored $\triangle$ and crossing only g-w edges can lead to another 3-colored $\triangle$ or else to a g-w edge on the g-w side of the simplex. However, these adjacent 3-colored $\triangle$s pair up (that is, there is an even number of them), and hence the total number of 3-colored $\triangle$s is in fact odd. ■

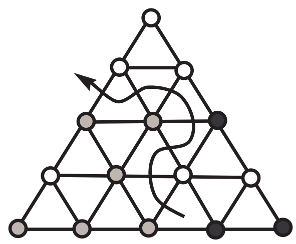

Figure 3: An example of a backwards walk that passes through multiple 3-colored $\triangle$s.

Claim The proof of Sperner's Lemma generalizes to higher dimensions.

This will not be proved here, however the idea is very much the same as for $k = 2$. Consider a side of the simplex. This side uses k colors, and so we can use the k dimensional version of the theorem. We get by the induction hypothesis that there are an odd number of k colored simplices on this side. Now we consider a walk through such k-colored simplices using exactly the given k colors. We claim as was done above that such walks pair up the k-colored side simplices, or lead to $k+1$ colored simplex in the larger dimension.

Proof (of Fixed Point Theorem) To start with we will color all nodes $x \in S_k$. Assume that $x = \sum \lambda_i e_i$ and $f(x) = \sum \mu_i e_i$. Assign x color i if $\lambda_i > \mu_i$ and this is the smallest such index (Note: if no such i exists, then $f(x) = x$ and we are done).

Now we want to use the Sperner lemma. Note first that the vertex e_i gets color i (as $\lambda_i = 1$, and all other values are 0). Furthermore, no points on a side of the simplex are colored by the opposite vertex. To see this note that a point x on the side opposite to vertex i has $\lambda_i = 0$, and hence cannot receive color i.

To use the Sperner lemma we need a subdivision of the simplex. We will consider a sequence of subdivisions. In the nth subdivision, each side of the simplex is subdivided into n segments. By the Sperner lemma, this subdivision has a multicolored simplex. Let x_n be the center of such a simplex. We want to argue that this sequence of points $x_1, x_2, \ldots$ has a limiting point that is a fixed point. First note that the x_i's are selected from a bounded and closed set (the simplex), so a subsequence of the sequence has a limit. Let x be such a limit.

We claim that $f(x) = x$. Suppose this is not the case, and let $x = \sum \lambda_i e_i$, and $f(x) = \sum \mu_i e_i$. If $f(x) \neq x$ then there is an index j such that $\mu_j > \lambda_j$. Let $\epsilon = \mu_j - \lambda_j$. For all points x' in a small enough neighborhood of x we must have $x' = \sum \lambda'_i e_i$ with $\lambda'_j < \lambda_j + \epsilon$. Furthermore, as f is continuous, for all x' in a small enough neighborhood of x we must have that $f(x') = \sum \mu'_i e_i$ with $\mu'_j > \mu_j - \epsilon$. Let δ be the smaller of the two neighborhoods above. We get that in the δ neighborhood of x there is no point colored j. However, this leads to a contradiction: The point x is the limit of centers of increasingly small simplices, and thus for high enough n in the sequence if x_n is in the subsequence its whole simplex is in the δ neighborhood of x. This is a contradiction, and therefore the simplex contains all colors.

Now consider multicolored simplices on finer and finer subdivisions. f is continuous and some subsequence has limit x. The claim is that $f(x) = x$. If x has a neighborhood on which this coordinate increases, $f(x)$ has coordinate i increased. The nodes on top of it want to have coordinates pushed down but the sum has to be 1 so that can't happen. As n gets bigger $x_1, x_2, ..., x_n, ...$ the triangulation gets smaller. ■