CS 6822, Fall 2011	Lecture 3
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Let G be a weighted undirected graph with edge weight $w(u,v) > 0$. The Laplacian of G is

$$L_G(u,v) = \begin{cases} -w(u,v) & \text{if } u \neq v \\ \sum_{u'\sim u} w(u,u') & \text{if } u = v \end{cases}$$

Laplacian and harmonic functions If $\phi \in \Re^{V}(G)$ and $L_G\phi = 0$, then

$$\phi(u) = \frac{\sum_{v\sim u} w(u,v)\phi(v)}{\sum_{v\sim u} w(u,v)} \qquad (*)$$

Suppose G is connected, finite, $(*)$ says for any u, $\phi(u)$ is weighted average of neighboring $\phi(v)$ values. This holds at $u \in \arg\max \phi(v)$. So all neighbors of u also have max value of ϕ. Same holds for their neighbors, etc. Thus, $\phi \equiv$ const.

Cor $L_G\phi = L_G\psi \Leftrightarrow \phi, \psi$ differ by an additive constant.

If $L_G\phi \equiv 0$, except at a few vertices ("boundary vertices"), we call ϕ harmonic with boundary values.

Harmonic functions and random walks Given connected, finite, weighted G, the random walk in G is defined such that the transition probability

$$P(u,v) = \frac{w(u,v)}{L_G(u,u)}.$$

The denominator $L_G(u,u)$ will be referred to as the *degree* of u and denoted by $\deg(u)$. It matches the standard definition of degree when egde weights are all equal to 1.

Ex1 $h(v) = \mathbb{P}(\text{RW starting from } v \text{ hits } s \text{ before } t)$

$L_G h(v)/\deg(v) =$ difference between $h(v)$ and average val of $h(v)$ after taking one RW step from v. This equals zero if $v \neq s,t$. Also $h(s) = 1, h(t) = 0$, and also $L_G h(s) + L_G h(t) = 0$

Ex2 $N(v) = \mathbb{E}[\#\text{ steps at } s \text{ before visit } t \text{ when start at } v]/\deg s$

$L_G N(v) = 0$ if $v \neq s,t$. $L_G N(s) = 1, L_G N(t) = -1$.

Conclusion $N = R_{\text{eff}} h$, where $R_{\text{eff}} := N(s)$

Proof. Let $r = \frac{1}{L_G h(s)} = -\frac{1}{L_G h(t)}$, then $L_G(rh - N) = 0$ everywhere. So

$$rh - N = \text{const}$$

Since $h(t) = N(t) = 0$, we know $rh = N$ and $r = \frac{N(s)}{h(s)} = N(s) = R_{\text{eff}}$. □

Lemma Random walk on G is transient iff for any vertex s, G has a function $0 \le \phi \le 1$ s.t.,

a) $\phi(s) = 0$ and $\exists v \ne s, \phi(v) > 0$

b) $L_G\phi = 0, \forall v \ne s$

Proof. If transient, $\phi(v) = \mathbb{P}(\text{RW starting at } v \text{ never reaches } s)$ works. For the other direction, given ϕ, must prove transient.

Take absorbing RW $s = v_0, v_1, \cdots =$ standard RW until first return to s (if any) and constant $v_{t+1} = \cdots = s$ after that. Then

$\forall t \ge 1, \mathbb{E}[\phi(v_{t+1})|v_t] = \phi(v_t)$ because

Case 1 $v_t \ne s, L_G\phi = 0$ at v

Case 2 $v_t = s, v_{t+1} = v_t = s$

For T large enough, we have $\mathbb{E}\phi(v_T) > 0$, say larger than ϵ, since in T steps, we can reach v from s. Then, by induction, we have $\mathbb{E}\phi(v_t) = \mathbb{E}\phi(v_T) > \epsilon$ for any $t > T$. This implies that $\mathbb{P}(v_t = s) < 1 - \epsilon$ for any $t > T$, so it's transient. □

Harmonic functions and electricity Suppose that the graph G is a network of electrical resistors of resistance $R(u, v)$. Let $C(u, v) = R(u, v)^{-1}$, a quantity called *conductance*. In an electrical network, if

$$\Delta(u) = \text{amount of external current flowing out at } u \text{ (or in, if negative)}$$

there is a current $i(u, v)$ on each edge (signed, so $i(u, v) = -i(v, u)$) and a potential (voltage) at each u satisfying,

(1) FLow conservation

$$\forall u, \sum_{v\sim u} i(u, v) = \Delta(u)$$

(2) Ohm's Law

$$\phi(u) - \phi(v) = i(u, v)R(u, v)$$

$$i(u, v) = C(u, v)(\phi(u) - \phi(v))$$

$\Rightarrow \forall u, \sum_{v\sim u} C(u, v)(\phi(u) - \phi(v)) = \Delta(u) \Rightarrow L_G\phi = \Delta$

$\therefore \phi$ uniquely determined, up to additive constants, by Δ

$\therefore i$ uniquely determined by Δ

$h(v) =$ Potential at v when we hook up a 1-volt battery to G with + term at s, ground at t

$N(v) =$ Potential at v when 1 unit of current flows from s to t and t is at 0 potential

$R_{\text{eff}} =$ pot. diff btw s, t when 1 unit of current flows from s to t through G

(Same as if G has a single resistor of resistance R_{eff} from s to t)

When RW starting at s is absorbed at t, how many times do we traverse $u \to v$? (in expectation)

Answer: Net number of traversals, counting $u \to v$ as +1 and $v \to u$ as -1, is $i(u, v)$.

Electrical flows and energy minimization Suppose i is an electrical flow in G with external sources Δ, d is a circulation, i.e.

$$\forall u, \sum_{v \sim u} d(u,v) = 0$$

Total energy dissipation of i vs $i+d$,

$$\begin{aligned}
&\sum_{(u,v)} (i(u,v) + d(u,v))^2 R(u,v)/2 \\
&= \sum_{(u,v)} i(u,v)^2 R(u,v)/2 + \underbrace{\sum_{(u,v)} d(u,v) R(u,v) i(u,v)}_{\sum_{(u,v)} (\phi(u)-\phi(v)) d(u,v) = 0} + \sum_{(u,v)} d(u,v)^2 R(u,v)/2 \\
&\geq \sum_{(u,v)} i(u,v)^2 R(u,v)
\end{aligned}$$

$\therefore$ Among all flows that satisfy flow conservation with external source Δ, the electrical flow minimizes energy dissipation. (Assume G finite, some results are true in infinite graphs that admit a finite energy flow.)