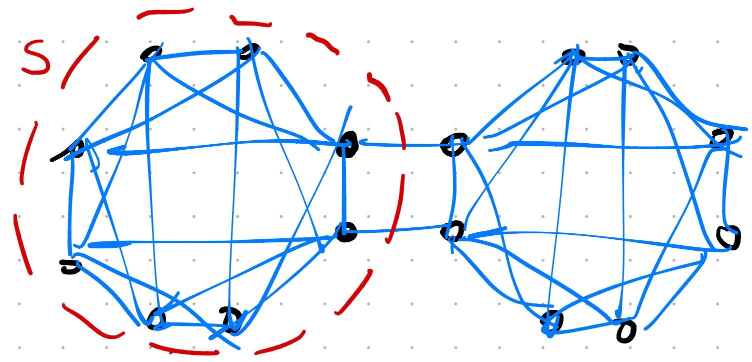
24 Nov 2025 Spectral graph partitioning

Recap.



$\partial S = \{$edges with endpoints in S and $V-S\}$.

$$y_v^S = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases}$$

$$\mathbb{E}_G\left[(y_u - y_v)^2\right] = \sum_{\{u,v\} \in \partial S} \frac{2c(u,v)}{d(V)} = \frac{2c(\partial S)}{d(V)}$$

$$\mathbb{E}_\pi\left[(y_u - y_v)^2\right] = \Pr_\pi(u \in S, v \notin S) + \Pr_\pi(v \in S, u \notin S) = 2\frac{d(S)}{d(V)} \cdot \frac{d(V-S)}{d(V)}$$

$$\lambda_2(L_G) = \min\left\{ \frac{\mathbb{E}_G\left[(y_u - y_v)^2\right]}{\mathbb{E}_\pi\left[(y_u - y_v)^2\right]} \;\middle|\; y \not\parallel \vec{1} \right\}$$

$$\leq \frac{\cancel{2}\, c(\partial S)/\cancel{d(V)}}{\cancel{2}\, d(S)\, d(V-S)/d(V)^{\cancel{2}}} = \frac{c(\partial S)\, d(V)}{d(S)\, d(V-S)}.$$

$$d(S) = \sum_{u \in S} d(u)$$

The sparsity of S is

$$\phi(S) = \frac{c(\partial S)\, d(V)}{d(S)\, d(V-S)}.$$

The expansion of S is

$$h(S) = \frac{c(\partial S)}{\min\{d(S), d(V-S)\}}$$

Lemma: $\forall S \neq \emptyset, V$

$$h(S) \leq \phi(S) \leq 2 \cdot h(S).$$

Proof: $$\phi(S) = \frac{c(\partial S)\, d(V)}{\min\{d(S), d(V-S)\} \cdot \max\{d(S), d(V-S)\}}$$

$$= h(S) \cdot \frac{d(S) + d(V-S)}{\max\{d(S), d(V-S)\}}$$

This factor between 1 and 2.

We saw $\lambda_2(\bar{L}_G) \leq \phi(S) \leq 2 \cdot h(S)$ so

$$\boxed{\frac{1}{2} \lambda_2(\bar{L}_G) \leq \min_S \{h(S)\}}$$

Cheeger's Inequality says

$$\boxed{\min_S \{h(S)\} \leq \sqrt{2 \lambda_2(\bar{L}_G)}}$$

i.e. if the 2nd eigenval of $\bar{L}_G$ is ε, the sparsest cut value is between $\frac{\varepsilon}{2}$ and $\sqrt{2\varepsilon}$.

In fact a set S obeying

$$h(S) \leq \sqrt{2\lambda_2(\overline{L}_G)}$$

can always be found by letting

$\vec{y}$ = eigenvector of $\overline{L}_G$ with eigenvalue $\lambda_2(\overline{L}_G)$

and

$$S = \{ u \mid y_u < \theta \}$$

for an appropriate choice of θ.

An example showing that $\lambda_2(\overline{L}_G) = \varepsilon$
$\min_S \{ h(S) \} = \Theta(\sqrt{\varepsilon})$ is possible
for $\varepsilon \to 0$.

$G = C_n$, cycle of length n.

[All edges capacity 1.]

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D_G = \begin{bmatrix} 2 & & & & \\ & 2 & & \text{\Large 0} & \\ & & 2 & & \\ & \text{\Large 0} & & \ddots & \\ & & & & 2 \end{bmatrix}$$

$$L_G = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & \text{\Large 0} & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & \text{\Large 0} & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ -1 & & & & & & -1 & 2 \end{bmatrix}$$

$$\overline{L}_G = \begin{bmatrix} 1 & -1/2 & & & & -1/2 \\ -1/2 & 1 & -1/2 & & \text{\Large 0} & \\ & -1/2 & 1 & & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & 1 & -1/2 \\ -1/2 & & & & -1/2 & 1 \end{bmatrix}$$

Eigenvectors $y^{(0)}, y^{(1)}, \ldots, y^{(n-1)}$

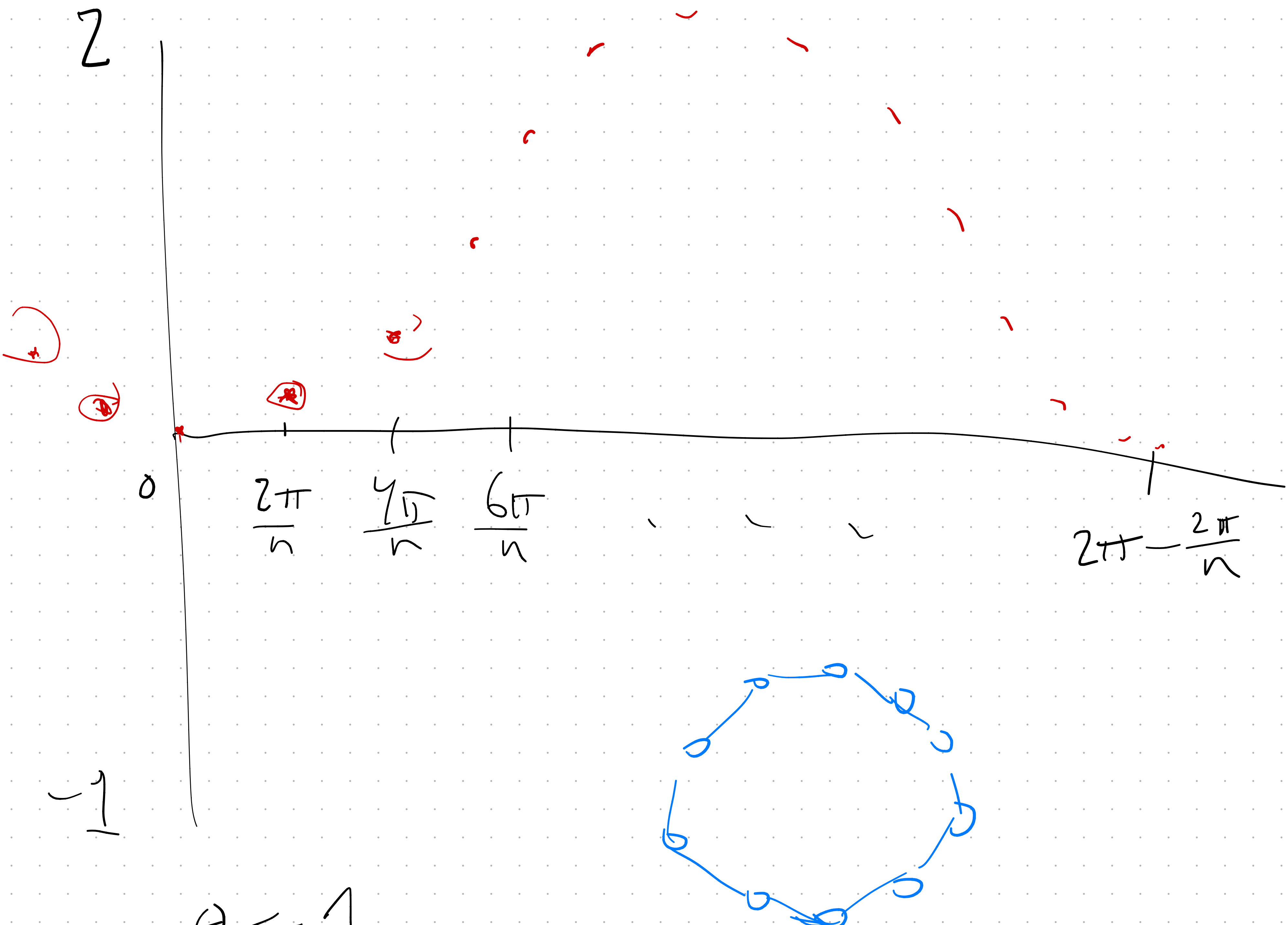
$$y_j^{(k)} = e^{2\pi i kj/n}$$

$$L_G y^{(k)} = y^{(k)} - \frac{1}{2} e^{\frac{2\pi i k}{n}} y^{(k)} - \frac{1}{2} e^{-\frac{2\pi i k}{n}} y^{(k)}$$

$$= \underbrace{\left[1 - \frac{1}{2} e^{\frac{2\pi i k}{n}} - \frac{1}{2} e^{-\frac{2\pi i k}{n}}\right]}_{\text{eigenvalue}} y^{(k)}$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

Eigenval of $y^{(k)} = 1 - \cos\left(\frac{2\pi k}{n}\right)$.



$$\theta \ll 1$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \dots$$

$$1 - \cos\theta \approx \frac{1}{2}\theta^2.$$

$$\lambda_2(L_G(C_n)) \simeq \frac{1}{2}\left(\frac{2\pi}{n}\right)^2 = \frac{2\pi^2}{n^2}.$$

$$\min_S \{h(S)\} = \frac{2}{n}$$

cut 2 edges

$\frac{n}{2}$ vertices each with degree 2