$\boxed{\text{21 Nov 2025}}$ Sparsity, expansion, Cheeger's ineq.

<u>Recap on self-adjoint eigenvalues.</u>

If V is n-dimensional vector space over $\mathbb{R}$

$\langle \cdot, \cdot \rangle$ is positive definite inner product

$A : V \to V$ is self-adjoint linear transformation

meaning $\forall_{x,y}$ $\langle x, Ay \rangle = \langle Ax, y \rangle$

then $\exists$ seq of eigenvalues $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$

and eigenvectors x_1 x_2 $\cdots$ x_n

s.t. (i) $\{x_i\}_{i=1}^n$ are <u>orthonormal</u>. $\langle x_i, x_j \rangle = \begin{cases} 1 & \text{if } i=j \\ \emptyset & \text{if } i \ne j \end{cases}$

(ii) x_i is an eigenvector with eigenvalue λ_i $\quad Ax_i = \lambda_i x_i$

(iii) x_i solves a constrained quadratic minimization

$$\lambda_i = \langle x_i, Ax_i \rangle = \min \left\{ \langle x, Ax \rangle \;\middle|\; \|x\| = 1,\ \langle x, x_j \rangle = 0 \ \forall_{j<i} \right\}$$

Recall $L_G = \begin{bmatrix} \ddots & & -c(u,v) \\ & \text{degrees} & \\ -c(u,v) & & \ddots \end{bmatrix}$

$\overline{L}_G = D_G^{-1} L_G = L_G$ with rows rescaled s.t. diagonal = all 1's.

L_G is symmetric.

$\overline{L}_G$ is self-adj pos def wrt $\langle \cdot, \cdot \rangle_d$

where $\langle x, y \rangle_d = \dfrac{\sum_{u \in V} d(u) x_u y_u}{\sum_{u \in V} d(u)}$ $= d(V)$.

This $\overline{L}_G$ has a probabilistic interpretation.

Let $\pi(u) = \dfrac{d(u)}{d(V)}$. π probab distrib on V.

Define two distribs on ordered pairs (u,v).

- PRODUCT DISTRIB $\mathbb{E}_{\pi}[f(u,v)]$

  Sample u, v independently from π.

- GRAPH DISTRIB $\mathbb{E}_G[f(u,v)]$

  Sample (u,v) with probability $\dfrac{c(u,v)}{d(V)}$.

<u>Lemma.</u> If $x \in \mathbb{R}^V$ then

(i) $\mathbb{E}_{\pi}[x_u] = \langle x, \vec{1} \rangle_d$

(ii) $\mathbb{E}_{\pi}[x_u^2] = \langle x, x \rangle_d$

(iii) $\frac{1}{2} \mathbb{E}_{\pi}[(x_u - x_v)^2] = \langle x, x \rangle_d - \left(\langle x, \vec{1} \rangle_d \right)^2$

(iv) $\frac{1}{2} \mathbb{E}_G[(x_u - x_v)^2] = \langle x, \overline{L}_G x \rangle_d$

Proof of Lemma.

$$\mathbb{E}_\pi[x_u] = \sum_u \pi(u) x_u = \frac{1}{d(V)} \sum_u d(u) x_u \cdot 1 = (x, 1)_d$$

$$\mathbb{E}_\pi[x_u^2] = \quad \ldots \quad = \frac{1}{d(V)} \sum_u d(u) x_u^2 = (x, x)_d$$

Part (iii): RHS = $\mathrm{Var}(x_u)$.

$x_u - x_v$ has expected value $\emptyset$.

$$\mathrm{Var}(x_u - x_v) = \mathbb{E}_\pi\left[(x_u - x_v)^2\right] = 2 \cdot \mathrm{LHS}$$

$$\| $$

$$\mathrm{Var}(x_u) + \mathrm{Var}(x_v) = 2 \cdot \mathrm{Var}(x_u) = 2 \cdot \mathrm{RHS}.$$

Proof of (iv):

$$\frac{1}{2} \mathbb{E}_G\left[(x_u - x_v)^2\right] = \frac{1}{2d(V)} \sum_{u,v} c(u,v)(x_u - x_v)^2$$

$$= \frac{1}{2d(V)} \sum_{u,v} c(u,v) x_u (x_u - x_v)$$

$$- \frac{1}{2d(V)} \sum_{u,v} c(u,v) x_v (x_u - x_v)$$

$$= \frac{1}{2d(V)} \sum_{u,v} c(u,v) x_u (x_u - x_v)$$

$$+ \frac{1}{2d(V)} \sum_{u,v} c(v,u) x_v (x_v - x_u)$$

$$= \frac{1}{d(V)} \sum_{u,v} c(u,v) x_u (x_u - x_v)$$

$$= \frac{1}{d(V)} \sum_{u} x_u \left(\sum_{v} c(u,v)(x_u - x_v) \right)$$

$$= \frac{1}{d(V)} \sum_{u} d(u) x_u \underbrace{\left(\frac{1}{d(u)} \sum_{v} c(u,v)(x_u - x_v) \right)}_{(D_G^{-1} L_G x)_u}$$

$$= \langle x, \overline{L}_G x \rangle_d$$

Recall $\lambda_1(L_G)=0$ because $L_G \vec{1} = 0$.

$$\lambda_2(L_G) = \min_{\substack{\|x\|_d=1, \\ \langle x, 1\rangle_d=0}} \left\{ \langle x, L_G x\rangle_d \right\}$$

$$= \min_{\substack{\langle y, 1\rangle_d = 0 \\ y \neq 0}} \left\{ \frac{\langle y, L_G y\rangle_d}{\langle y, y\rangle_d} \right\}$$

$$= \min_{\substack{\mathbb{E}_{\pi}[y_u]=0 \\ y \neq 0}} \left\{ \frac{\frac{1}{2}\mathbb{E}_G[(y_u-y_v)^2]}{\frac{1}{2}\mathbb{E}_{\pi}[(y_u-y_v)^2] + (\mathbb{E}_{\pi}(y_u))^2} \right\}$$

$$= \min_{\substack{\mathbb{E}_{\pi}[y_u]=0 \\ y \neq 0}} \left\{ \frac{\frac{1}{2}\mathbb{E}_G[(y_u-y_v)^2]}{\frac{1}{2}\mathbb{E}_{\pi}[(y_u-y_v)^2]} \right\}$$

$$= \min_{y \nparallel \vec{1}} \left\{ \frac{\frac{1}{2}\mathbb{E}_G[(y_u-y_v)^2]}{\frac{1}{2}\mathbb{E}_{\pi}[(y_u-y_v)^2]} \right\}$$

Now suppose G has a sparse cut

meaning S such that

$S, V-S$ are non-empty and

$$\underbrace{\sum_{u \in S} \sum_{v \notin S} c(u,v)}_{= c(\partial S)} \ll d(S) \cdot d(V-S).$$

Define y^S by $y^S_u = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{if } u \notin S. \end{cases}$

$$\mathbb{E}_G\left[(y_u - y_v)^2\right] = \frac{2\, c(\partial S)}{d(V)}.$$

$$\mathbb{E}_\pi\left[(y_u - y_v)^2\right] = \Pr_\pi(u \in S, v \in V-S) + \Pr_\pi(u \notin S, v \in S)$$

$$= 2 \cdot \frac{d(S)}{d(V)} \cdot \frac{d(V-S)}{d(V)}$$

Conclusion: sparse cut $\Rightarrow$ $\lambda_2(L_G) \ll 1$.