$\boxed{\text{19 Nov 2025}}$

Normalized Laplacian, sparsity, expansion

$G = (V, E)$ undirected

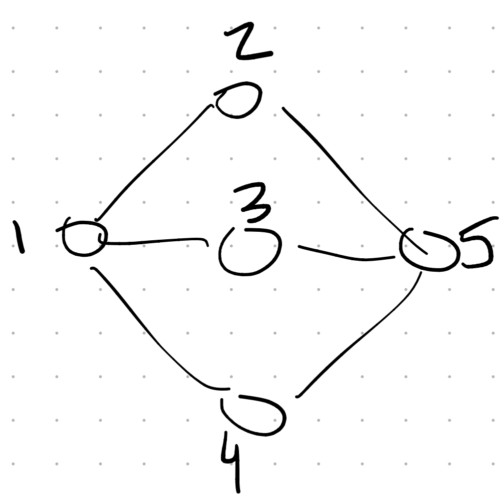
Symmetric edge capacity matrix $c(u,v)$

$$(A_G)_{uv} = c(u,v)$$

$$(D_G)_{uv} = \begin{cases} 0 & \text{if } u \neq v \\ \sum_{w \neq u} c(u,w) & \text{if } u = v \end{cases}$$

$$L_G = D_G - A_G$$

$$\bar{L}_G = D_G^{-1} \cdot L_G$$




$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad D = \begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 3 \end{bmatrix}$$

All capacities 1.

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix} \quad \bar{L} = \begin{bmatrix} 1 & -1/3 & -1/3 & -1/3 & 0 \\ -1/2 & 1 & 0 & 0 & -1/2 \\ -1/2 & 0 & 1 & 0 & -1/2 \\ -1/2 & 0 & 0 & 1 & -1/2 \\ 0 & -1/3 & -1/3 & -1/3 & 1 \end{bmatrix}$$

If $x \in \mathbb{R}^V$ and $L_G x = y$ then

$$y_u = \frac{1}{d(u)} \cdot \sum_{v \neq u} c(u,v)\,(x_u - x_v)$$

$$= x_u - \frac{\sum_{v \neq u} c(u,v)\, x_v}{\sum_{v \neq u} c(u,v)}$$

"normalized Laplacian represents the operation of subtracting the avg. of neighbors' values."

Recall: $\langle x, L_G x \rangle = \sum_{(u,v)} c(u,v)\,(x_u - x_v)^2$

"unnormalized Laplacian represents evaluating a quadratic energy function that penalizes diff. in endpoints' labels."

Define the degree-averaging inner product:

$$\langle x, y \rangle_d = \sum_{v \in V} \frac{d(v)}{d(V)}\, x_v y_v$$ where

$$d(V) = \sum_{v \in V} d(v).$$

This $\langle \cdot, \cdot \rangle_d$ is a positive definite inner product.

i.e., a binary operation $\mathbb{R}^V \times \mathbb{R}^V \to \mathbb{R}$ that is symmetric, linear in each variable, $\langle x, x \rangle = 0 \iff x = \vec{0}$; else $\langle x, x \rangle > 0$.

Fact. $\overline{L}_G$ satisfies $\langle x, \overline{L}_G y \rangle_d = \langle \overline{L}_G x, y \rangle_d$

"self-adjoint"

$$\langle x, y \rangle_d = \frac{1}{d(V)} x^T D_G y$$

$$\langle x, \overline{L}_G y \rangle_d = \frac{1}{d(V)} x^T \cancel{D_G} (\cancel{D_G^{-1}} L_G) y$$

$$\langle \overline{L}_G x, y \rangle_d = \frac{1}{d(V)} (\overline{L}_G x)^T D_G y$$

$$= \frac{1}{d(V)} x^T \overline{L}_G^T D_G^T y$$

$$= \frac{1}{d(V)} x^T (D_G \overline{L}_G)^T y$$

$$= \frac{1}{d(V)} x^T (\cancel{D_G} \cancel{D_G^{-1}} L_G)^T y$$

$$= \frac{1}{d(V)} x^T L_G y$$

self-adjoint w.r.t. dot product $\Longleftrightarrow$ symmetric

$\forall x, y \quad \langle x, Ay \rangle = \langle Ax, y \rangle$

$$\sum x_i a_{ij} y_j \qquad\qquad \sum a_{ij} x_j y_i$$

Theorem. If $A$ is a symmetric matrix then its eigenvals are all real,

$$\lambda_{min}(A) = \min_{\|x\|=1} \{ \langle x, Ax \rangle \}$$

$$\lambda_{max}(A) = \max_{\|x\|=1} \{ \langle x, Ax \rangle \}.$$

More generally, for any pos def inner product if we redefine $\|x\|$ to mean $\sqrt{\langle x, x \rangle}$ we can say: a matrix self-adjoint w.r.t. inner prod has only real eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ and their eigenvectors $x_1, \dots, x_n$ are orthonormal w.r.t. $\langle , \rangle$.

$\forall k \in [n]$ $\quad \lambda_k$ = minimum value of $\langle x, Ax \rangle$ over $\Downarrow$

$x_k$ = minimizer of $\langle x, Ax \rangle$ over

$\{ x : \|x\| = 1, \langle x, x_i \rangle = 0 \ \forall i < k \}$.

Back to $\overline{L}_G$ which is self-adj

w.r.t. $\langle\ ,\ \rangle_d$

$$\langle x, \overline{L}_G x\rangle_d = \frac{1}{d(V)} x^T L_G x$$

$$= \frac{1}{d(V)} \sum_{u \neq v} c(u,v)\left(x_u - x_v\right)^2 \geq 0$$

$\lambda_1(\overline{L}_G) = 0$, $\quad x_1 = \vec{1}$.

$\lambda_2(\overline{L}_G) = ??$, $\quad \langle x_2, x_1\rangle = 0$

$$\| \quad \min\left\{ \langle x, \overline{L}_G x\rangle_d \;\middle|\; \langle x, \vec{1}\rangle = 0,\ \|x\|_d = 1 \right\}$$

<u>Next up:</u> When $G$ has a sparse cut

we can use it to find $x$

satisfying $\langle x, \vec{1}\rangle = 0$, $\|x\|_d = 1$, $\langle x, \overline{L}_G x\rangle_d \ll 1$.

<u>Then</u>. When $\lambda_2(\overline{L}_G) \ll 1$, show how thresholding

$x_2$ leads to finding a sparse cut.