17 Nov 2025

Finishing multicommodity flow

Introducing the graph Laplacian.

Announcements

1. Problem Set 4 on Canvas.
   Due Fri., 12/5, 11:59pm
2. Final proj announcement on Ed.
   Due Mon., 12/15

P1 (maximizer) "edge inspector"

Choose an edge e. Inspect how much flow is crossing e.

P2 (minimizer) "router"

Choose paths $Q = (P_1, \ldots, P_k)$. Send $\frac{1}{k}$ units of flow on each path.

Payoffs: $a_{eQ} = \dfrac{n_e(Q)}{k}$

$$\min_{x \in \Delta^n} \max_{y \in \Delta^m} \{x^T A y\} = \max_y \min_x \{x^T A y\}$$

$\underline{\text{CLAIM}}$, The $\underline{\text{minimax value}}$ of this game is $\frac{1}{k \cdot r^*}$ where r^* is the max concurrent flow rate.

$\underline{\text{PROOF}}$. First suppose f^* is a concurrent MCF of rate r^*. Then $f^* = (f^*_Q)_{Q \in \mathcal{Q}}$ and $r^* = \sum_{Q \in \mathcal{Q}} f^*_Q$.

So if we define $x_Q = \frac{f^*_Q}{r^*}$ then $x \in \Delta^{\mathcal{Q}}$.

Feasibility: $\forall e \quad \sum_Q f^*_Q \cdot n_Q(e) \leq 1$.

$$\forall e \quad \sum_Q x^*_Q \frac{n_Q(e)}{k} \leq \frac{1}{k \cdot r^*}.$$

Summary: $\min_{x\in\Delta^{\mathcal{Q}}} \max_{y\in\Delta^{E}} \{x^T A y\}$

$= V(\text{GAME}) \leq \frac{1}{k\cdot r^*}$.

Opposite ineq: to show $v(\text{GAME}) \geq \frac{1}{k r^*}$

equiv'tly $r^* \geqslant \frac{1}{k\cdot v(\text{GAME})}$. ✓

Suppose x^* is an opt strategy for router. i.e.

$$\sum_Q x^*_Q = 1, \Longrightarrow \sum_Q f_Q = \frac{1}{k\cdot v(\text{GAME})}$$

$$\forall e \quad \sum_Q x^*_Q \cdot n_Q(e)/k \leq v(\text{GAME})$$

$$\sum_Q x^*_Q \cdot n_Q(e) \leqslant k\cdot v(\text{GAME})$$

$$\sum_Q f_Q \cdot n_Q(e) \leq 1$$

Define $f_Q = \frac{x^*_Q}{k\cdot v(\text{GAME})}$. $\Longrightarrow f_Q$ feasible

Recall: If we have a sequence of T strategy pairs (i_t, j_t) in this case (q_t, Q_t)

where $n_{ij} = \#\{t : (i_t, j_t) = (i, j)\}$

then we can define regret of players 1 and 2

$$R_1 = \frac{\max_i \left(\sum_t A[i, j_t]\right) - \sum_t A[i_t, j_t]}{T}$$

$$R_2 = \frac{\sum_t A[i_t, j_t] - \min_j \sum_t A[i_t, j]}{T}$$

We proved

$$\left| \frac{\sum_t A[i_t, j_t]}{T} - v(\text{GAME}) \right| \leq R_1 + R_2$$

Regret(Hedge with N strategies)

$$\leq \sqrt{\frac{2 \ln(N)}{T}}$$ ← to make $< \varepsilon$

$$T > \frac{2 \ln N}{\varepsilon^2}.$$

Instead P1 (edge inspector) runs Hedge over {edges}.

P2 waits for P1 to commit to mixed strategy in round t then best responds.

$$R_1 \leq \sqrt{\frac{2 \log N}{T}}$$

$$R_2 \leq 0$$

$$\left| \frac{1}{T} \sum A[i_t, j_t] - \frac{1}{k \cdot r^*} \right| < \sqrt{\frac{2 \log N}{T}}$$

To get mult. $(1+\varepsilon)$-approx to r^*

need RHS $< \frac{\varepsilon}{k r^*}$.

Solve for T:

$$\sqrt{\frac{2 \log N}{T}} < \frac{\varepsilon}{k r^*}$$

\# edges

Use $r^* \geq \frac{1}{k}$ ... $\sqrt{\frac{2 \log N}{T}} < \varepsilon$

suffices, $T > \frac{2 \log N}{\varepsilon^2}$.

The Laplacian of a graph G.

G = ~~undirected~~ graph. $= (V, E)$

$c : V^2 \longrightarrow \mathbb{R}_{\geq 0}$ $\left[c(u,v) = 0 \text{ when } \{u,v\} \notin E \right]$

$c(u,v) = c(v,u)$.
$c(v,v) = 0$

Define $d(v) = \sum_u c(u,v)$.

(Matches degree when c is $\{0,1\}$-valued.)

Some matrices assoc. to G.

$A_G[u,v] = c(u,v)$ // adjacency

$D_G[u,v] = \begin{cases} d(v) & \text{when } u = v \\ 0 & \text{when } u \neq v \end{cases}$ // diagonal

$L_G = D_G - A_G$. // Laplacian.

Def. The Laplacian quadratic form of G is the function on $\mathbb{R}^V$ defined by

$$Q(x) = x^T L_G x.$$

$$= \sum_{u,v} x_u L_G[u,v] x_v$$

$$= \sum_v d(v) \cdot x_v^2 - \sum_v \sum_{u \neq v} x_u c(u,v) x_v$$

$$= \sum_v \sum_{u \neq v} c(u,v) \cdot x_v^2 - c(u,v) \cdot x_u \cdot x_v$$

$$= \sum_v \sum_{u \neq v} c(u,v)(x_v - x_u) x_v$$

$$= \sum_{\{u,v\}} c(u,v) \left[(x_v - x_u) \cdot x_v + (x_u - x_v) \cdot x_u \right]$$

$$= \sum_{\{u,v\}} c(u,v) (x_v - x_u)^2 \geq 0$$

$\Longrightarrow$ L_G is PSD symm matrix

$\Longrightarrow$ All eigenvals ≥ 0.

Nullspace of L_G

$= \{$ vectors x constant on each conn compt of $G \}$