$\boxed{\text{14 Nov 2025}}$ Finishing Mult Weights for Multicommodity Flow

RECAP. 2 player zero sum game.

Payoff matrix $A = (a_{ij}) \in \mathbb{R}^{n \times m}$

Player 1 chooses $x \in \Delta^n = \{ \sum x_i = 1, \ x_i \geq 0 \ \forall i \}$

" " 2 " " $y \in \Delta^m$

P1 goal: $\max \ x^T A y = \sum_{i,j} a_{ij} x_i y_j$

P2 goal: $\min \ x^T A y$

Von Neumann:

$$\max_{x \in \Delta^n} \min_{y \in \Delta^m} (x^T A y) = \min_{y \in \Delta^m} \max_{x \in \Delta^n} (x^T A y)$$

Suppose the game is played T times and

$n_{ij} \triangleq$ # times P1 chooses i while P2 chooses j.

Define: $x_i = \frac{1}{T} \sum_j n_{ij}$ // fraction of times P1 chose i

$y_j = \frac{1}{T} \sum_i n_{ij}$ // fraction of times P2 chose j

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ $N_{ij} = \frac{n_{ij}}{T}$

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i \qquad R_1 = \max_i \left(e_i^T A y \right) - \mathrm{Tr}(N^T A)$$ // P1's "regret"

$$R_2 = \mathrm{Tr}(N^T A) - \min_j \left(x^T A e_j \right)$$ // P2's regret

Then observe

$$\left(\max_i e_i^T A y\right) - R_1 = \mathrm{Tr}(N^T A) = \left(\min_j x^T A e_j\right) + R_2$$

$$\left(\max_{\tilde{x} \in \Delta^n} \tilde{x}^T A y\right) - R_1 \qquad\qquad \min_{\tilde{y} \in \Delta^m} \left(x^T A \tilde{y}\right) + R_2$$

$$\vee \qquad\qquad \wedge$$

$$\boxed{\min_{\tilde{y} \in \Delta^m} \max_{\tilde{x} \in \Delta^n} \left(\tilde{x}^T A \tilde{y}\right)} - R_1 \qquad\qquad \boxed{\max_{\tilde{x} \in \Delta^n} \min_{\tilde{y} \in \Delta^m} \left(\tilde{x}^T A \tilde{y}\right)} + R_2$$

val_1(P2 moves first) $\geq$ val_1(P1 moves first)

val_1(P1 moves first) $\leq$ val_1(P2 moves first) $\leq$ val_1(P1 first) $+ R_1 + R_2$

If $R_1(T), R_2(T) \to 0$ as $T \to \infty$

one obtains von Neumann's Theorem.

Relating multicommodity flow to game theory....

__P1__ (maximizer) "edge inspector"

Choose an edge e. Inspect how much flow is crossing e.

__P2__ (minimizer) "router"

Choose paths $Q = (P_1, \ldots, P_k)$.
Send $\frac{1}{k}$ units of flow on each path.

__Payoffs.__ $$a_{eQ} = \frac{n_e(Q)}{k}.$$