3 Nov 2025 — Finishing Sparsest Cut

Announcements.

1. Prelim 2 covers {max flow, min cut, apx algs}
   10/6 — 10/27 lectures
2. Prelim 2 room assignments: to be announced on Ed.
3. Prelim 2 make-up... Mon 11/10 5-6pm (?)
4. Prob Set 1 grades to be released tmrw.

(P)

$$\max \quad z$$

$$\text{s.t.} \quad \sum_{p \in P(s_i,t_i)} x_p \geq z \quad \forall i \in [k]$$

$$\sum_{p:\, e \in p} x_p \leq c(e) \quad \forall e \in E$$

$$x_p \geq 0$$

(D)

$$\min \quad \sum c(e)\, y_e$$

$$\text{s.t.} \quad \sum_{e \in p} y_e \geq d_i \quad \forall i \in [k],\ \forall p \in P(s_i,t_i)$$

$$\sum_{i=1}^{k} d_i \geq 1$$

$$y_e, d_i \geq 0$$

Randomized rounding of fractional to integer cuts...

Given: $\vec{y} = (y_e)_{e \in E}$ satisfying constraints of (D)

Goal: Find edge set $A \subseteq E$ with sparsity(A) not much greater than $\sum c(e) y_e$.

distribution TBD

Procedure.

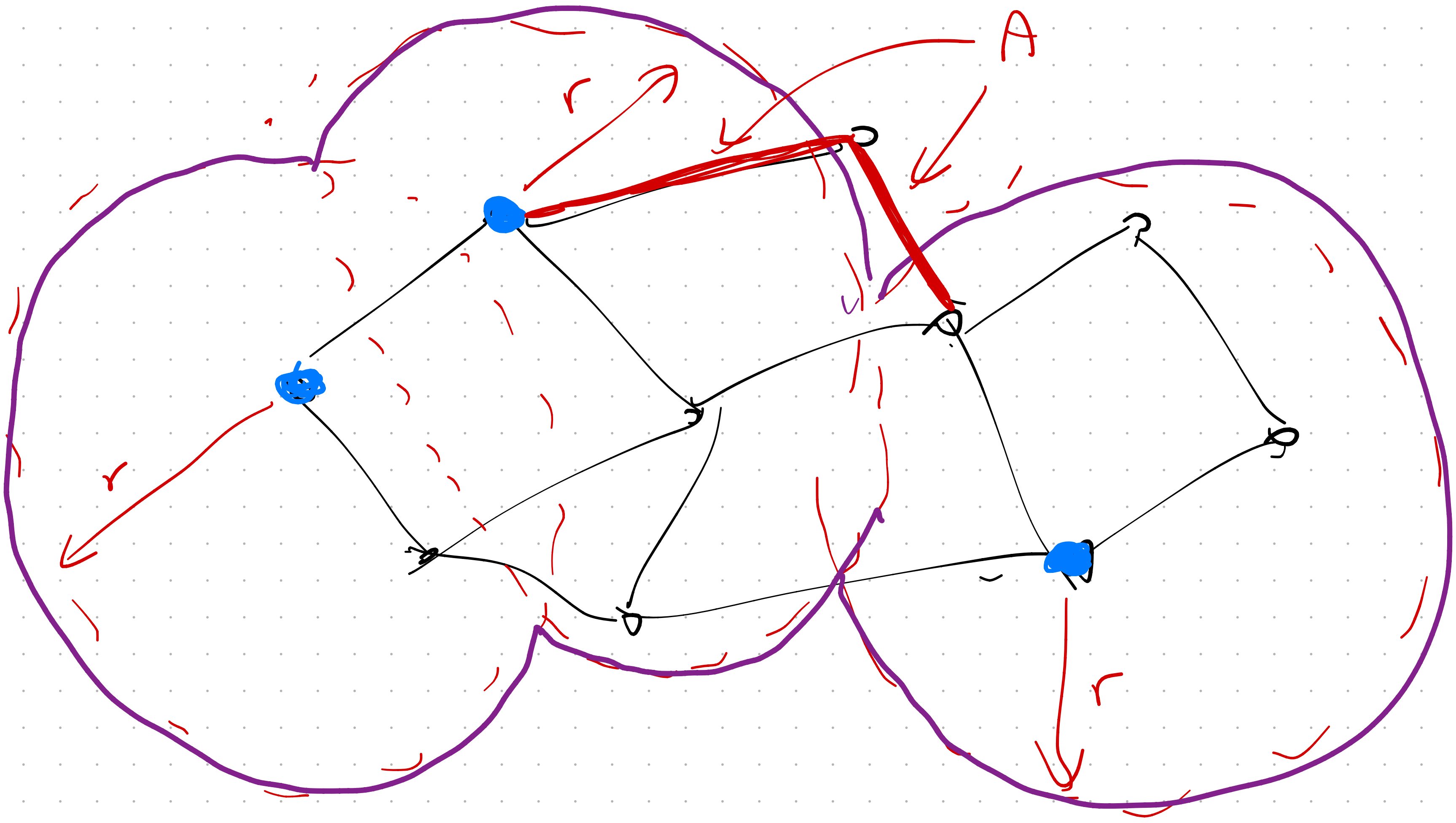
1. Sample random set $W \subseteq \{s_1, t_1, \ldots, s_k, t_k\}$.
2. Sample "search radius" $r \in [0,1]$ uniformly random, indep't of W.
3. For all $v \in V$ let

$$d(v, W) = \min_{w \in W} \{ d(v,w) \}$$

shortest path length w.r.t. edge lengths $\vec{y}$.

4. Let

$$A = \left\{ (u,v) \;\middle|\; \begin{array}{c} d(u,W) < r < d(v,W) \\ \text{or} \\ d(v,W) < r < d(u,W) \end{array} \right\}$$



$$\text{sparsity}(A) = \frac{\text{cap}(A)}{\text{sep}(A)}$$

Will show: $\mathbb{E}[\text{cap}(A)] \leq \sum c(e)\, y_e$

$$\mathbb{E}[\text{sep}(A)] \geq \frac{c}{\log(2k)}$$

for constant c.

These imply

$$\sum_A \Pr(A) \cdot \text{cap}(A) \leq \frac{\sum c(e) y_e}{c} \cdot \log(2k) \cdot \sum_A \Pr(A) \cdot \text{sep}(A)$$

$$\implies \exists A \quad \text{cap}(A) \leq \frac{\sum c(e) y_e}{c} \cdot \log(2k) \cdot \text{sep}(A)$$

$$\text{sparsity}(A) \leq O(\log k) \cdot \left(\sum c(e) y_e\right)$$

$$\mathbb{E}[\mathrm{cap}(A)] = \sum_e c(e) \cdot \Pr(e \in A)$$

$$= \sum_{e=(u,v)} c(e) \cdot \Pr(r \text{ lands btw } d(u,W) \text{ \& } d(v,W))$$

$$\leq \sum_{e \in (u,v)} c(e) \cdot |d(u,W) - d(v,W)|$$

$$\leq \sum_{e \in (u,v)} c(e)\, d(u,v) \leq \sum_e c(e)\, y_e .$$

To reason about sep(A):

Edge set A will separate s_i from t_i

whenever $0 < r < \frac{1}{2} d(s_i, t_i)$