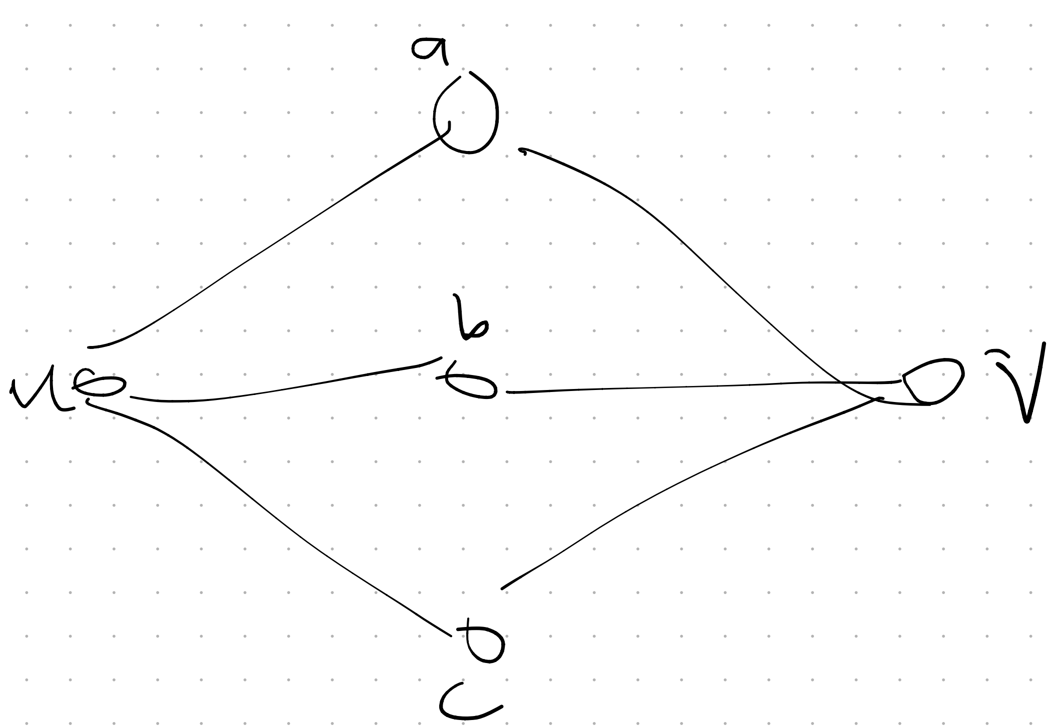
31 Oct 2025 Approx. Max-Flow Min-Cut

Recall: Max Concurrent flow asks for max rate r_{MCF} at which we can simultaneously route flow from s_1 to $t_1, \ldots, s_k$ to t_k.

Sparsest cut asks for an edge set, A, whose removal disconnects at least one $s_i - t_i$ pair, and minimizes

$$\frac{(\text{capacity of edges cut})}{(\#\text{ of pairs discon})}$$

$$\begin{pmatrix}\text{max concurrent} \\ \text{flow rate}\end{pmatrix} \le \begin{pmatrix}\text{min cut} \\ \text{sparsity}\end{pmatrix}$$



All edges capacity 1.

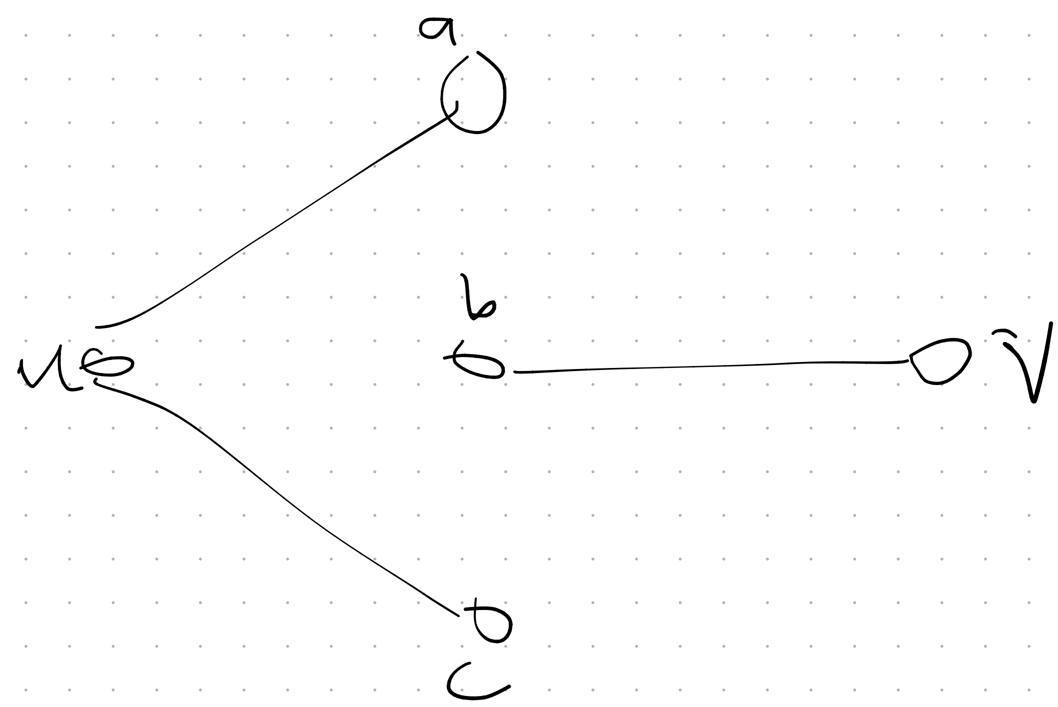
Terminal pairs

$u \longrightarrow v$

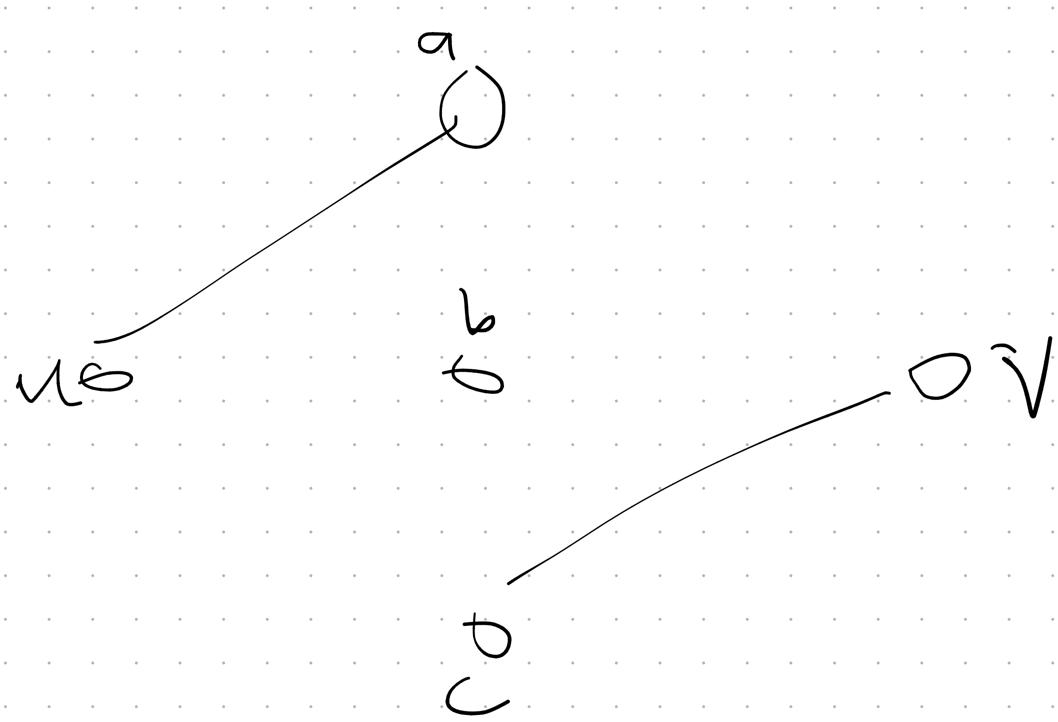
$a \longrightarrow b$

$b \longrightarrow c$

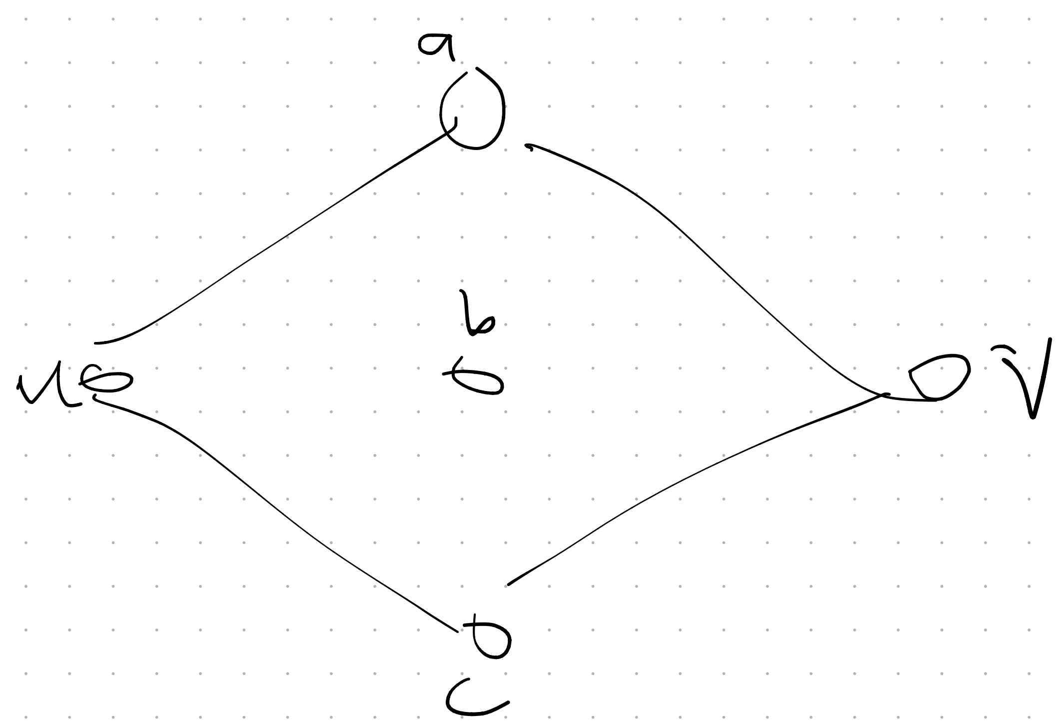
$c \longrightarrow a$



sparsity = 1



Sparsity = 1



Sparsity = 1

Def A "fractional cut" is any assignment of a length y_e to each edge e such that for at least one $i \in [k]$

$$\boxed{d_y(s_i, t_i)} > 0$$

length of shortest path from s_i to t_i interpreting edge lengths as y_e.

$$cap(y) = \sum_e c_e y_e$$

$$sep(y) = \sum_{i=1}^{k} d_y(s_i, t_i)$$

$$sparsity(y) = \frac{cap(y)}{sep(y)}$$

A "normalized fractional cut"

is one with $sep(y) = 1$,

Every fractional cut is equivalent to a normalized one, up to scaling, and they have the same sparsity.

SPARSEST NORMALIZED FRACTIONAL CUT:

$$\min \sum_e c_e y_e$$

z

$$\text{s.t.} \quad \sum_{i=1}^{k} d_i = 1$$

x_P

$$\sum_{e \in P} y_e - d_i \geq 0 \qquad \forall i \quad \forall P \in \mathcal{P}(s_i, t_i)$$

$$y_e, d_i \geq 0$$

$$\max \quad z$$

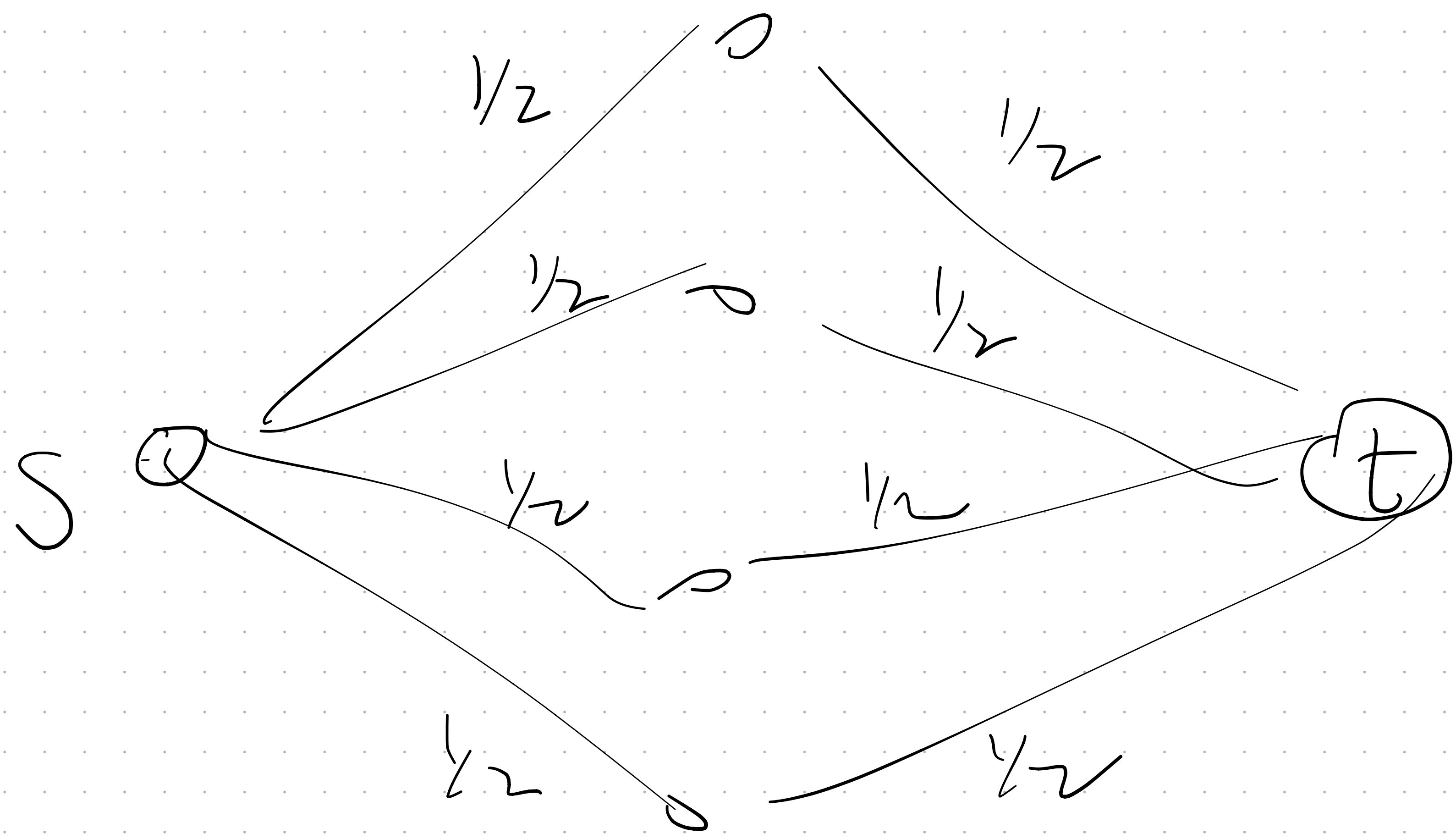
$$\text{s.t.} \quad \sum_{i=1}^{k} \sum_{\substack{p \in \mathcal{P}(s_i,t_i) \\ e \in p}} x_p \leq c_e \quad \forall e$$

$$z - \sum_{p \in \mathcal{P}(s_i,t_i)} x_p \leq 0 \quad \forall i$$

$$x_p, z \geq 0$$

$$\left(\begin{array}{c}\text{max concur.} \\ \text{flow}\end{array}\right) = \left(\begin{array}{c}\text{sparsest} \\ \text{fract. cut}\end{array}\right)$$

$$\overset{???}{\approx} \left(\begin{array}{c}\text{Sparsest} \\ \text{cut}\end{array}\right)$$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