$\boxed{\text{29 Oct 2025}}$

# Multicommodity Flow and Sparsest Cut

Story Line:

- Multicommodity flow is computationally easy. (More on that next week...)

- Graph partitioning problems like sparsest cut are usually computationally hard. Approximation algorithms with apx. factor $O(\log k)$ usually exist, where $k$ = # terminals being partitioned, due to approximate max flow – min cut theorems for multi-commodity problems.

Ingredients:

$G$ = (directed or) undirected graph

$c : E \to [0, \infty]$ capacities

$\{(s_i, t_i)\}_{i=1}^{k}$ pairs of (source, destination) that want to be connected.

$P(s_i, t_i) = \{\text{paths in } G \text{ from } s_i \text{ to } t_i\}$

$\mathcal{Q} = P(s_1, t_1) \times \cdots \times P(s_k, t_k)$

$= \{\text{tuples of paths connecting all terminals}\}$

A feasible multicommodity flow is a vector $f = (f_p)_{p \in \bigcup_{i=1}^{k} P(s_i, t_i)}$

satisfying $f_p \geq 0 \quad \forall p$

$$\sum_{p \in \bigcup_{i=1}^{k} P(s_i, t_i) \,:\, e \in p} f_p \leq c(e) \quad \forall e \in E.$$

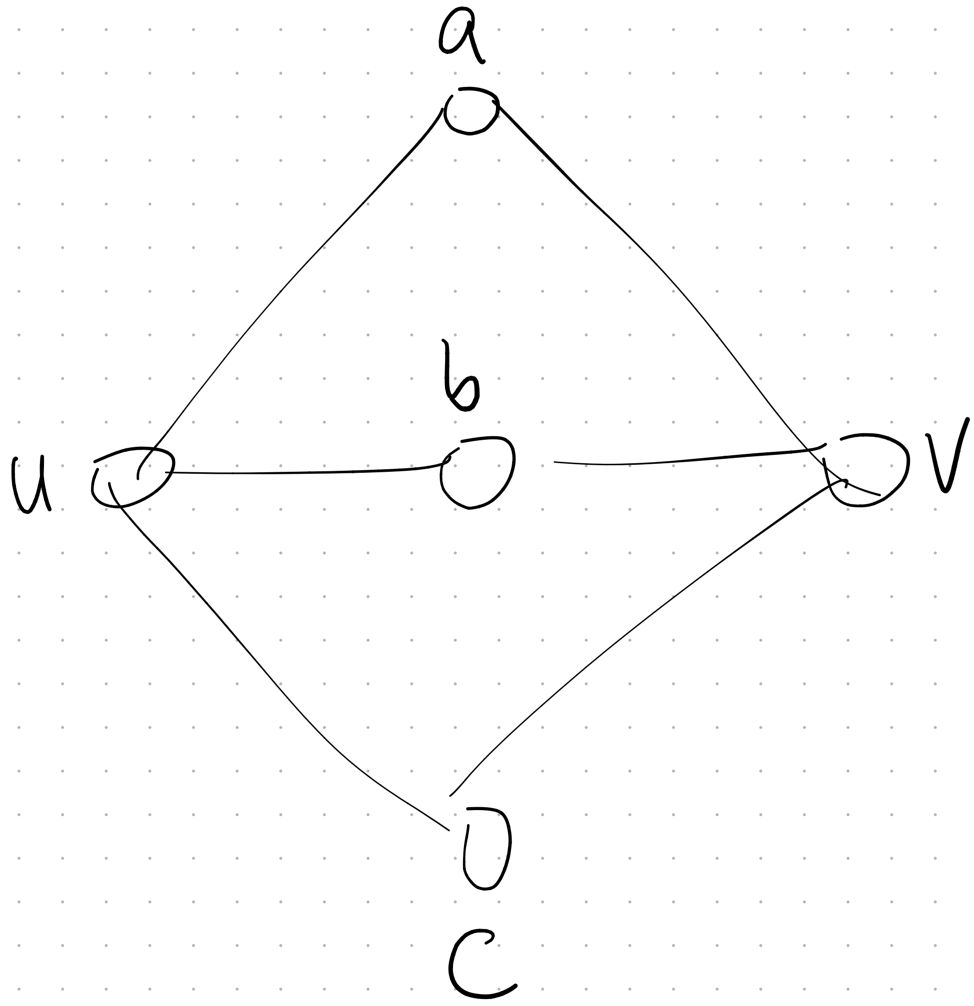
A feasible concurrent flow is

$f = (f_Q)_{Q \in \mathcal{Q}}$ satisfying

$$f_Q \geq 0 \qquad \forall Q$$

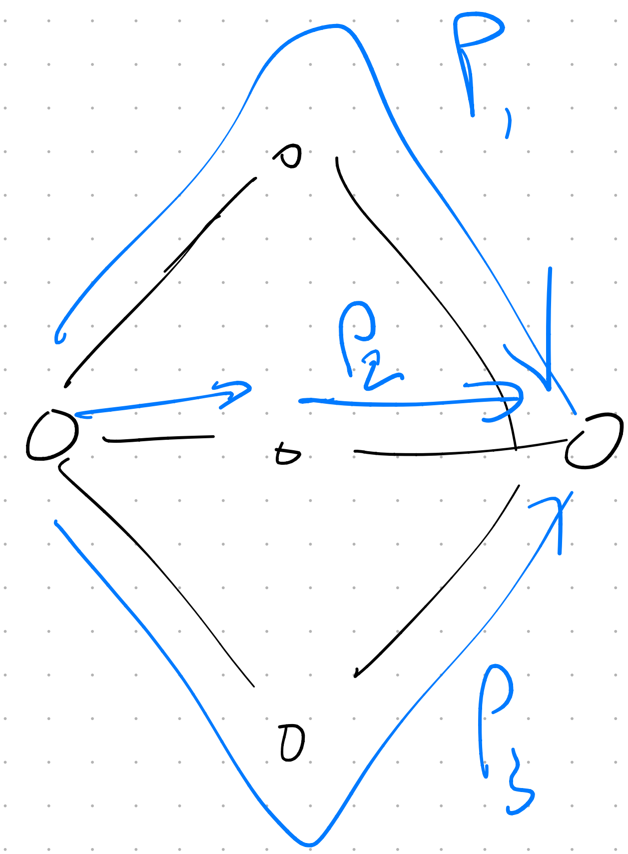
$$\sum_{Q \in \mathcal{Q}} \boxed{n_Q(e)} \cdot f_Q \leq c(e) \quad \forall e \in E$$

# of paths in $Q$ that contain $e$.
$0 \leq n_Q(e) \leq k$.



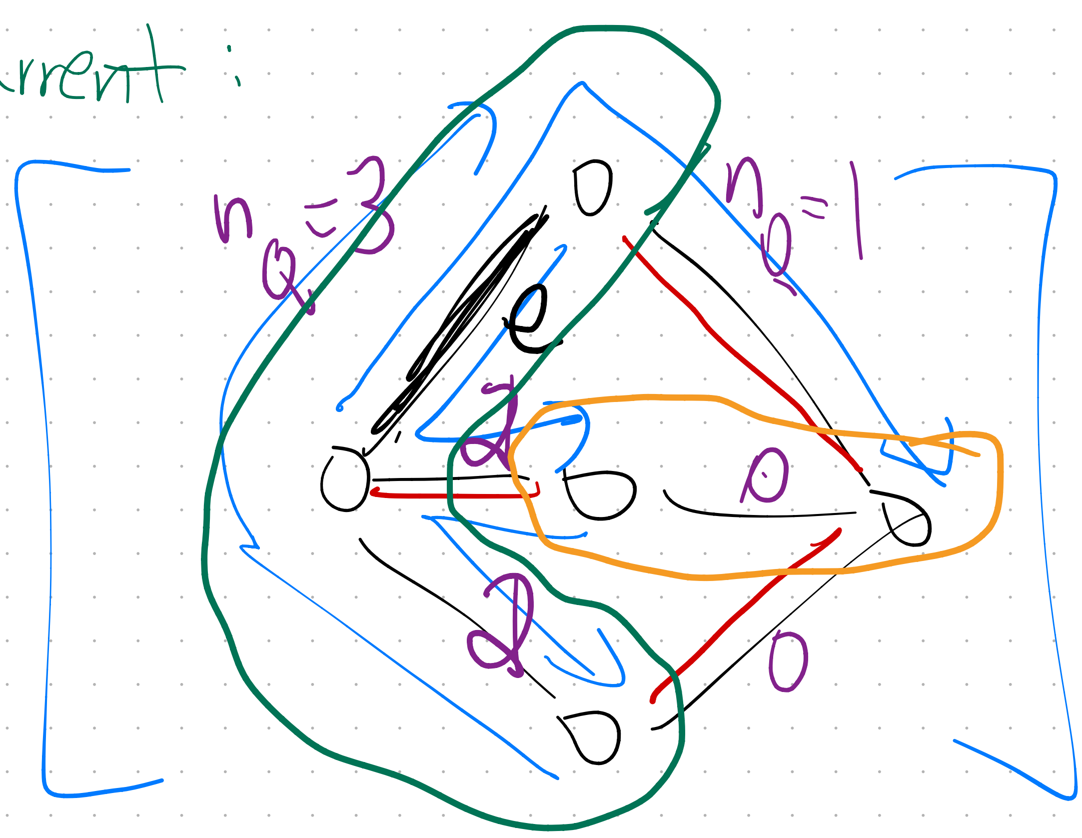

$$\{\text{Terminal pairs}\} = \left\{ \begin{array}{c} (a,b) \\ (b,c) \\ (c,a) \\ (u,v) \end{array} \right\}$$

Feasible MCF:




Feasible concurrent:

$\frac{1}{8}$




$+ \frac{1}{8}$ [ 5 other path-tuples ]

$$\sum_Q n_Q(e) \cdot f_Q = \frac{1}{8}(3) + \frac{2}{8}(2) + \frac{1}{8}(1)$$
$$= \frac{3}{8} + \frac{4}{8} + \frac{1}{8} = \underline{1}.$$

This is a feasible concurrent flow of <u>rate</u> $6/8$.

(In general rate $= \sum_Q f_Q$).

A "sparse cut" can serve as a readily verifiable upper bound on max concurrent flow.

Cut $\approx$ set of edges, $A$.

$$\mathrm{cap}(A) = \sum_{(u,v)\in A} c(u,v)$$

$$\mathrm{sep}(A) = \#\{\, i \mid s_i, t_i \text{ are in different compt's of } G\setminus A \,\}$$

$$= \#\{\, i \mid \forall P \in \mathcal{P}(s_i, t_i),\ P \cap A \neq \emptyset \,\}$$

$$\mathrm{Sparsity}(A) = \frac{\mathrm{cap}(A)}{\mathrm{sep}(A)}$$

MCF rate $r$ requires

$$\mathrm{cap}(A) \geq r \cdot \mathrm{sep}(A)$$

$$\frac{\mathrm{cap}(A)}{\mathrm{sep}(A)} \geq r$$

Sparsest cut value

$$\min \left\{ \mathrm{Sparsity}(A) \;\middle|\; \mathrm{sep}(A) > 0 \right\} \geq$$

max concurrent flow rate.