27 Oct 2025

Weighted Set Cover & Vertex Cover Approximation via LP Duality

Set Cover. n elements to be covered, $[n] = \{1, \dots, n\}$

$S_1, \dots, S_m \subseteq [n]$ $\quad \bigcup_{j=1}^{m} S_j = [n]$

$w(1), \dots, w(m) \geq 0$

Goal. Find J s.t. $\bigcup_{j \in J} S_j = [n]$,

minimize $w(J) = \sum_{j \in J} w(j)$.

Greedy algorithm. In iteration $t = 1, 2, \dots$

let n_{t-1} be # elements not covered during iterations up to $t-1$.

Choose j_t to minimize

$$\frac{w(j_t)}{\text{\# new elements covered by } S_{j_t}}$$

(SC-LP) $\quad \min \sum_{j=1}^{m} w(j)\, x_j$

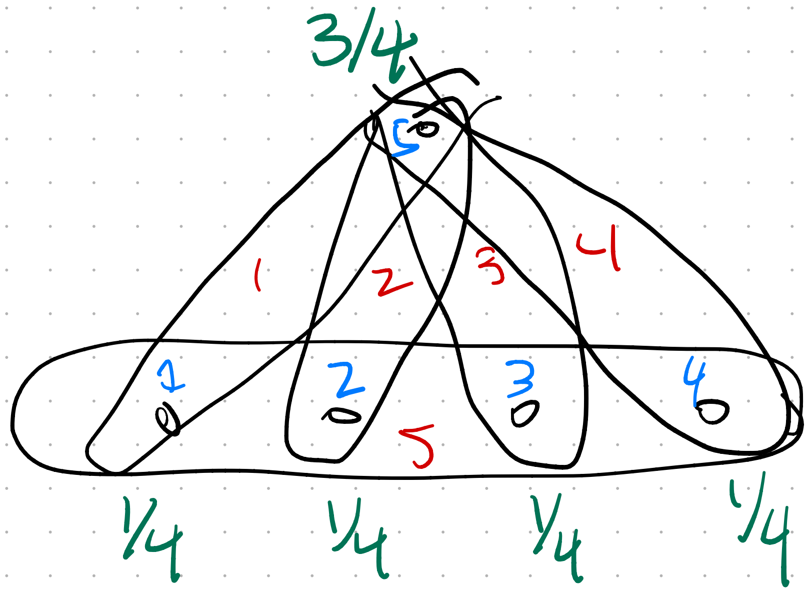
s.t. $\sum_{j:\, i \in S_j} x_j \geq 1 \quad \forall\, i \in [n] \quad (y_i)$

$x_j \geq 0$

(SC-DUAL) $\quad \max \sum_{i=1}^{n} y_i$

s.t. $\sum_{i \in S_j} y_i \leq w(j) \quad \forall\, j = 1, \dots, m$

$y_i \geq 0$



All sets have weight 1.

$$\min \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \vec{x}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \vec{x} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x} \geq 0$$

Analyzing GREEDY.

$J = \emptyset$

$\mathcal{C} = \{\text{uncovered elements}\} = [n]$

while $\mathcal{C} \neq 0$:

choose j to minimize $\dfrac{w(j)}{\#(S_j \cap \mathcal{C})}$

$J = J \cup \{j\}$

$\mathcal{C} = \mathcal{C} \setminus S_j$

$\tilde{y}_i = \dfrac{w(j)}{\#(S_j \cap \mathcal{C})}$ for all $i \in S_j \cap \mathcal{C}$

elements being freshly covered

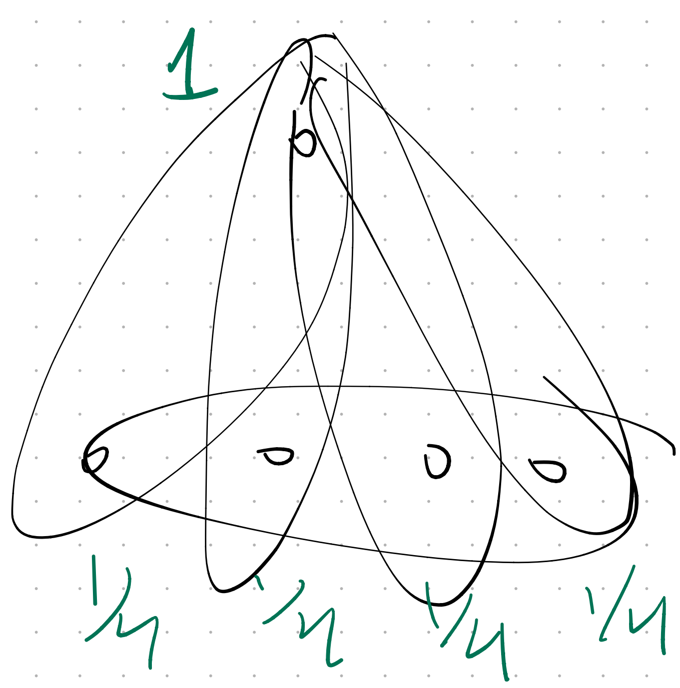
endwhile

output J

By design, $\sum_{i=1}^{n} \tilde{y}_i = \sum_{j \in J} w(j)$

$= w(\text{GREEDY}).$

If $(\tilde{y}_1, \ldots, \tilde{y}_n)$ were feasible for (SC-DUAL) we would have found a dual solution whose value equals $w(\text{GREEDY})$.



To scale down $\tilde{y}$ to a dual-feasible y, we need to know: how large can

$\sum_{i \in S_j} \tilde{y}_i$ be compared to $w(j)$?

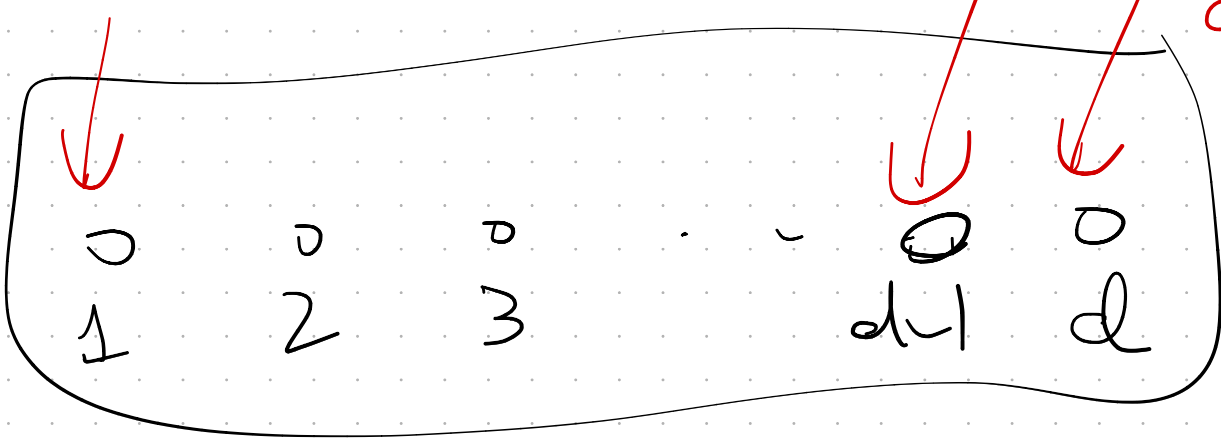
Order element of S_j from latest-covered to earliest covered.

$d = |S_j|$

$\tilde{y}_1 \leq w(j)$

$\tilde{y}_{d-1} \leq \frac{w(j)}{d-1}$

$\tilde{y}_d \leq \frac{w(j)}{d}$



$$\sum_{i=1}^{d} \tilde{y}_i \leq w(j) \cdot \left(1 + \frac{1}{2} + \dots + \frac{1}{d} \right)$$

$$\leq w(j) \left[1 + \ln d \right]$$

So, $y = \tilde{y} \cdot \left[1 + \ln\left(\max_j |S_j| \right) \right]^{-1}$

then y is feasible for (SC-DUAL).