24 Oct 2025

Approximating Vertex Cover & Set Cover

Modifying randomized alg for $w(S)$ objective...

INPUT: Graph $G=(V,E)$, weight function $w: E \to \mathbb{R}_+$

$S = \emptyset$

while G has an uncovered edge (u,v):

sample u w/ prob $p_u =$?? — see below for formula

sample v w/ prob $p_v = 1 - p_u =$??

add the sampled vertex into S

endwhile

output S

Try for: $\mathbb{E}\, w(S \cap S^*) \geq \mathbb{E}\, w(S \setminus S^*)$

$u \in S^*, v \notin S^*$ $\quad + p_u w(u) \geq + p_v w(v)$

$u \in S^*, v \in S^*$ $\quad + p_u w(u) + p_v w(v) \geq 0$ ✓

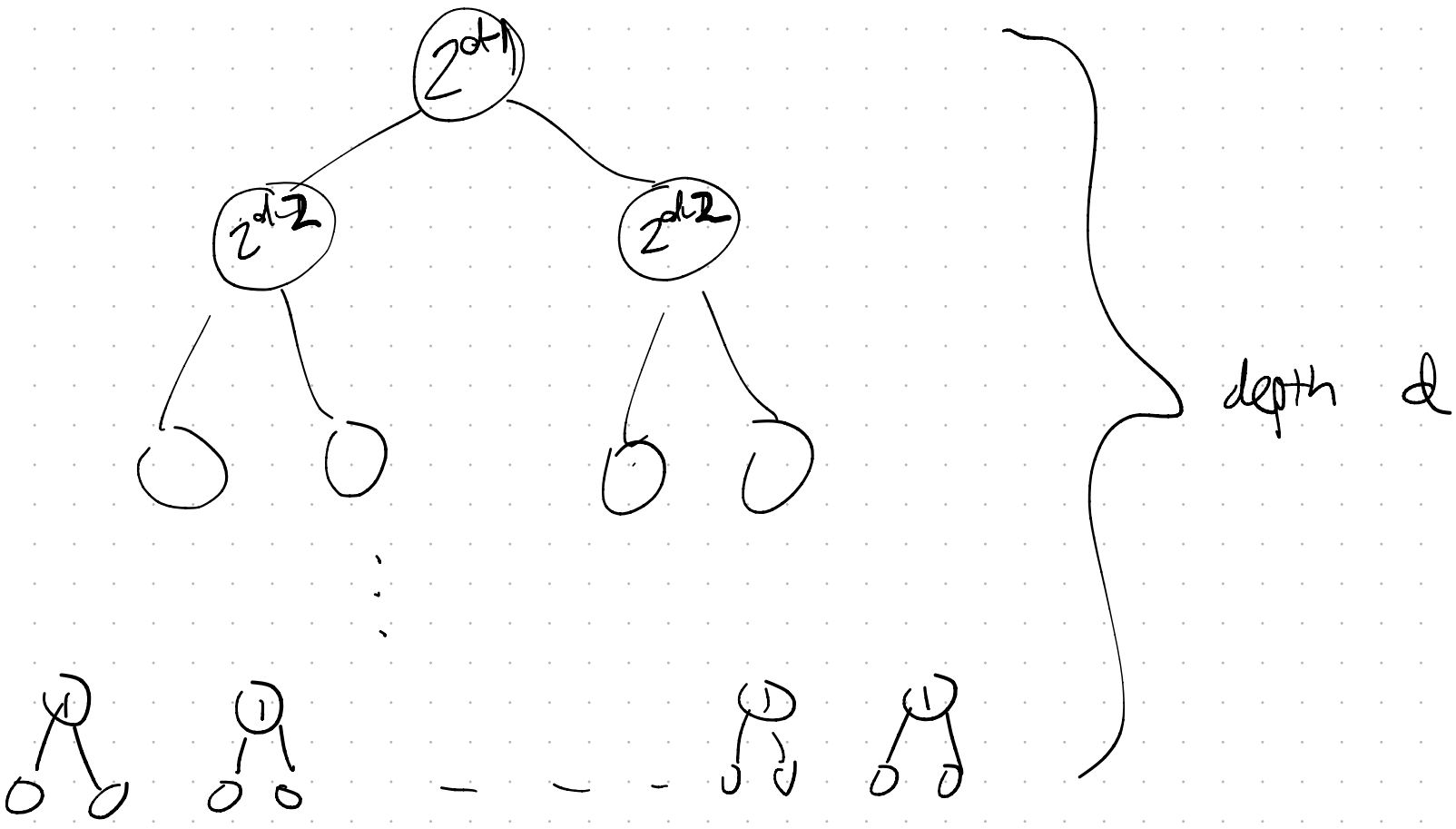
$u \notin S^*, v \in S^*$ $\quad + p_v w(v) \geq + p_u w(u)$

$$p_u w(u) = p_v w(v)$$

$$p_u + p_v = 1$$

$$\Rightarrow \quad p_u = \frac{w(v)}{w(u)+w(v)} \qquad p_v = \frac{w(u)}{w(u)+w(v)}.$$

$$\mathbb{E}\, w(S) = \mathbb{E}\, w(S \cap S^*) + \mathbb{E}\, w(S \setminus S^*) \leq 2\, \mathbb{E}\, w(S \cap S^*) \leq 2 w(S^*).$$



LEAVES

Copies of internal nodes

Connect a leaf to every copy of every ancestor.

$\#L = 2^d$

$\#R = d \cdot 2^{d-1}$

OPT is this

GREEDY chooses this vertex cover

SET COVER

Given: Elements $[n] = \{1, \dots, n\}$

Subsets $S_1, S_2, \dots, S_m \subseteq [n]$

Weights $w(1), \dots, w(m) \geq 0$

Problem: Choose J such that $\bigcup_{j \in J} S_j = [n]$

Minimize $w(J) = \sum_{j \in J} w(j)$.

(Unweighted set cover: the case when $w(j) = 1 \quad \forall j$)

GREEDY (UNWEIGHTED) SET COVER:

$J = \emptyset$

while $\exists$ uncovered elements

choose j s.t. S_j has max # of uncovered elements.

add j into J

mark all elements of S_j as covered

endwhile

output J.

PROP. GREEDY is a $\ln(n)$-approx. to unweighted set cover.

PROOF. Let n_t = # of elements uncovered at end of iteration t.

Suppose $J_{OPT} = \{j_1, j_2, \ldots, j_k\}$. ← # sets in OPT solution.

Claim: $n_{t+1} \leq \left(1 - \frac{1}{k}\right) n_t$.

Starting from $n_0 = n$,

$$n_t \leq \left(1 - \frac{1}{k}\right)^t \cdot n.$$

$1 - x \leq e^{-x}$

$$\leq e^{-t/k} \cdot n$$

When $t > k \ln(n)$, $e^{-t/k} < \frac{1}{n}$,

so $n_t < 1 \Longrightarrow n_t = 0$.

For weighted set cover, GREEDY means:

$J = \emptyset$

while $\exists$ uncovered elements

for all j define $GAIN(j) =$ # uncoverd elts in S_j

choose j that minimizes $\frac{w(j)}{GAIN(j)}$.

(avoiding j s.t. $GAIN(j) = 0$)

add j into J, mark S_j as covered.

endwhile