22 Oct 2025

Approximation Algorithms for Vertex Cover

2-approximation algorithm for vertex cover:

an algorithm that always selects a valid vertex cover, and its size is at most 2× (min vertex cover size).

GREEDY (max degree first, then delete the covered edges and iterate) surprisingly doesn't work.

A 2-approx algorithm

1. Pick any <u>maximal</u> (set-wise) matching, M.
2. Let $S = \{\text{endpoints of edges in } M\}$. Output S as the vertex cover.

$$|S| = 2 \cdot |M|$$

$$|\text{min vertex cover}| \geq |M|$$

A randomized 2-approx. alg:

- always outputs valid vertex cover.
- expected size $\leq 2 \cdot$ (min vtx cov size).

$S = \emptyset$

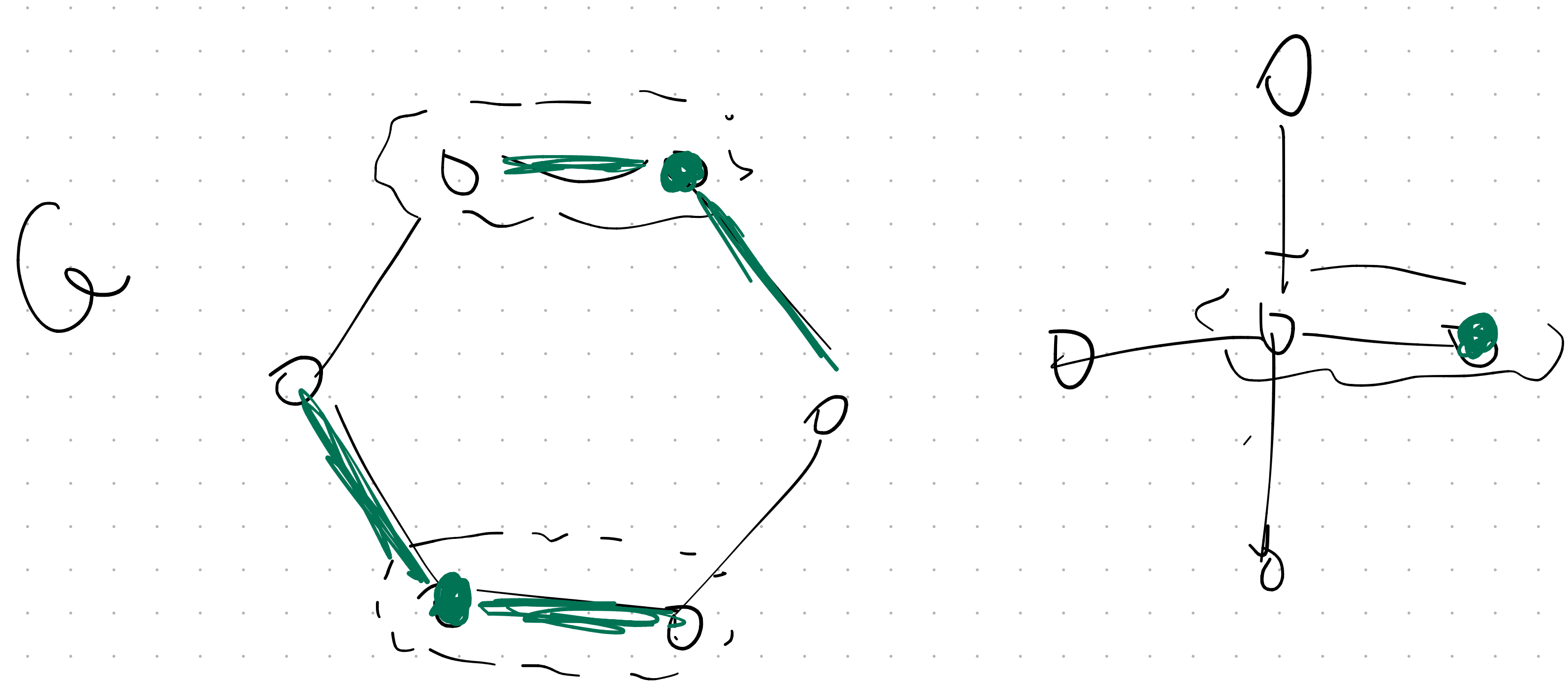
while G has an edge (u,v) not covered by S:

    sample u or v with equal probability

    add the sampled vertex to S.

endwhile

output S.

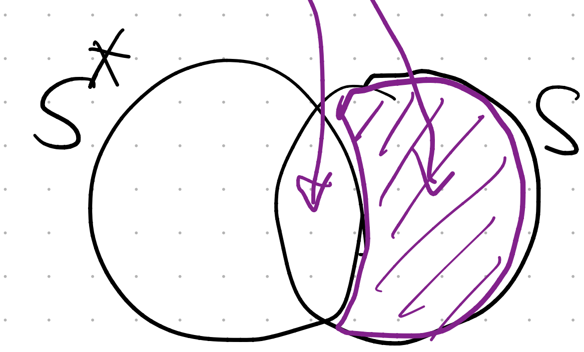


Why is this a 2-approx?

Let S^* = min vertex cover

S = random vtx cover we chose.

<u>Will prove</u>: $\mathbb{E}\,\#(S \cap S^*) \geq \mathbb{E}\,\#(S \setminus S^*)$



$$\mathbb{E}\,\#S = \mathbb{E}\,\#(S \cap S^*) + \mathbb{E}\,\#(S \setminus S^*)$$
$$\leq 2 \cdot \mathbb{E}\,\#(S \cap S^*) \leq 2 \cdot |S^*|$$

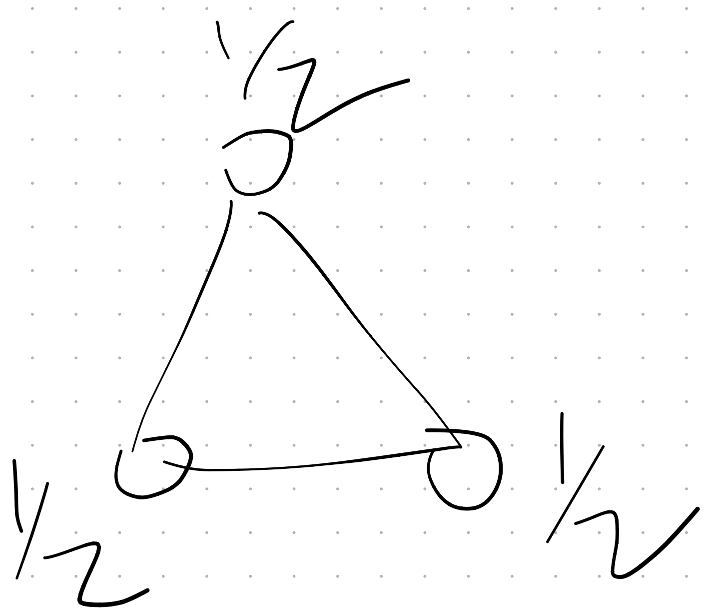
<u>Weighted Vertex Cover</u>.

Choose vertex set S which covers all edges. Minimize $\sum_{v \in S} w(v)$ where $\{w(v) \geq 0\}$ are given.

$$\min \sum_{v \in V(G)} w(v)\, x_v \qquad \text{(VC-LP)}$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall (u,v) \in E(G)$$

$$x_v \geq 0 \quad \forall v \in V(G).$$



Solve (VC-LP) to obtain $\vec{x}$.

Set $S = \{ v \mid x_v \geq \frac{1}{2} \}$.

If S^* a min-weight vertex cover

let x_S, x_{S^*} are $\{0,1\}$ vectors corresponding to S and S^*

and x_{LP} = OPT solution to VC-LP.

$$w(S^*) = \sum_{v \in V} w(v)\, x_{S^*}(v)$$

$$\geq \sum_{v \in V} w(v)\, x_{LP}(v)$$

$$\geq \frac{1}{2} \cdot \sum_{v \in V} w(v)\, x_S(v)$$

$$= \frac{1}{2} w(S).$$