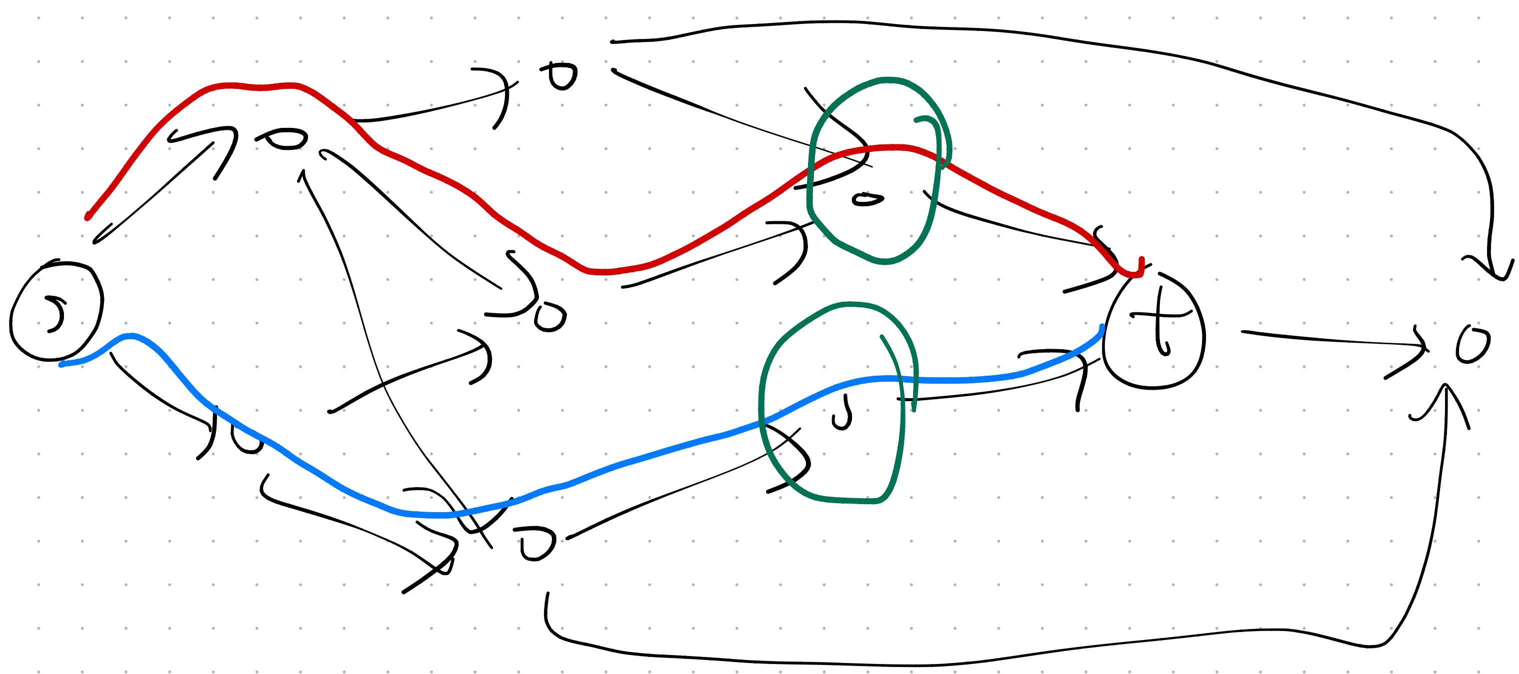
$\boxed{\text{20 Oct 2025}}$ Applications of Max Flow – Min Cut

Menger's Theorem(s):

In a $\begin{pmatrix}\text{directed}\\ \text{undirected}\end{pmatrix}$ graph G with vertices s & t, the max # of $\begin{bmatrix}\text{edge-disjoint}\\ \text{internally vertex-disjoint}\end{bmatrix}$ paths from s to t equals the min # of $\begin{bmatrix}\text{edges}\\ \text{vertices} \notin \{s,t\}\end{bmatrix}$ that must be removed from G to disconnect s from t.



Proof for edge-disj paths in directed graphs:

Make G into a flow network by $c_{uv} = 1$ $\forall$ edge (u,v).

max flow in (G,c) = min cut capacity in (G,c)

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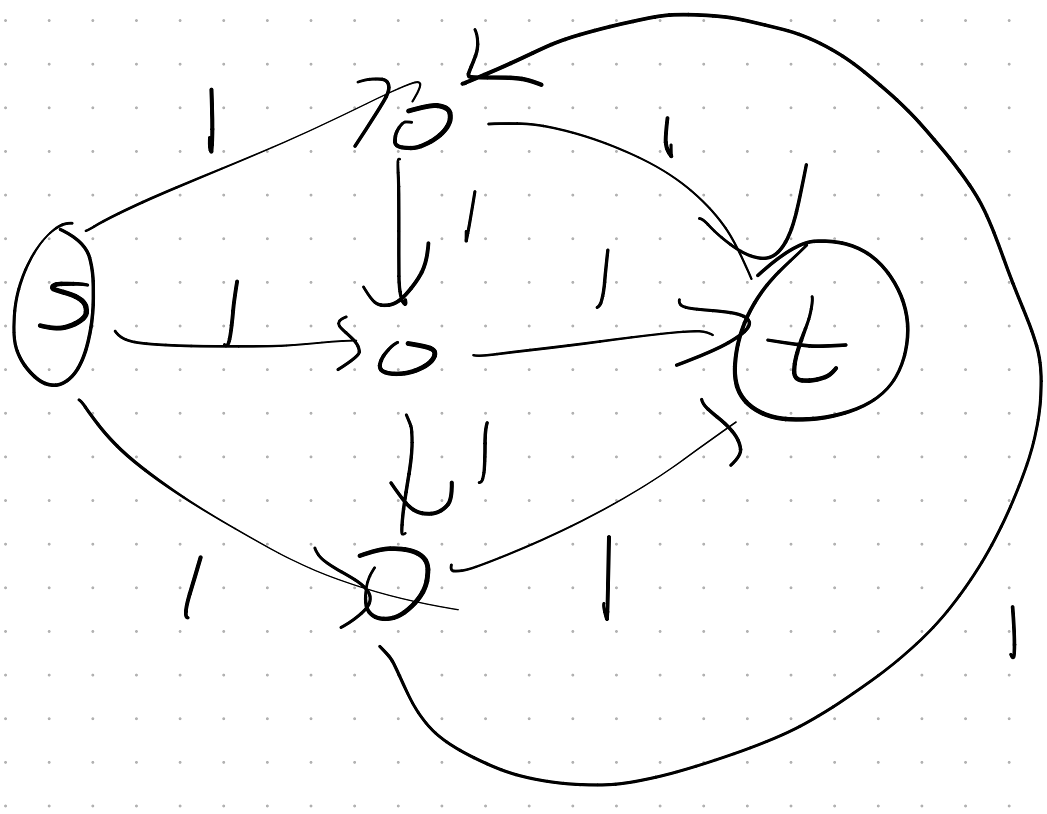
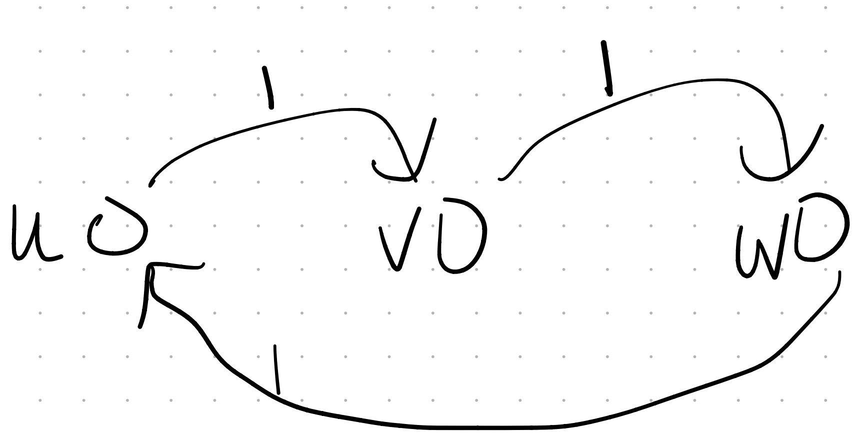
max # edge disj s-t paths

min # of edge deletions to disconnect s from t



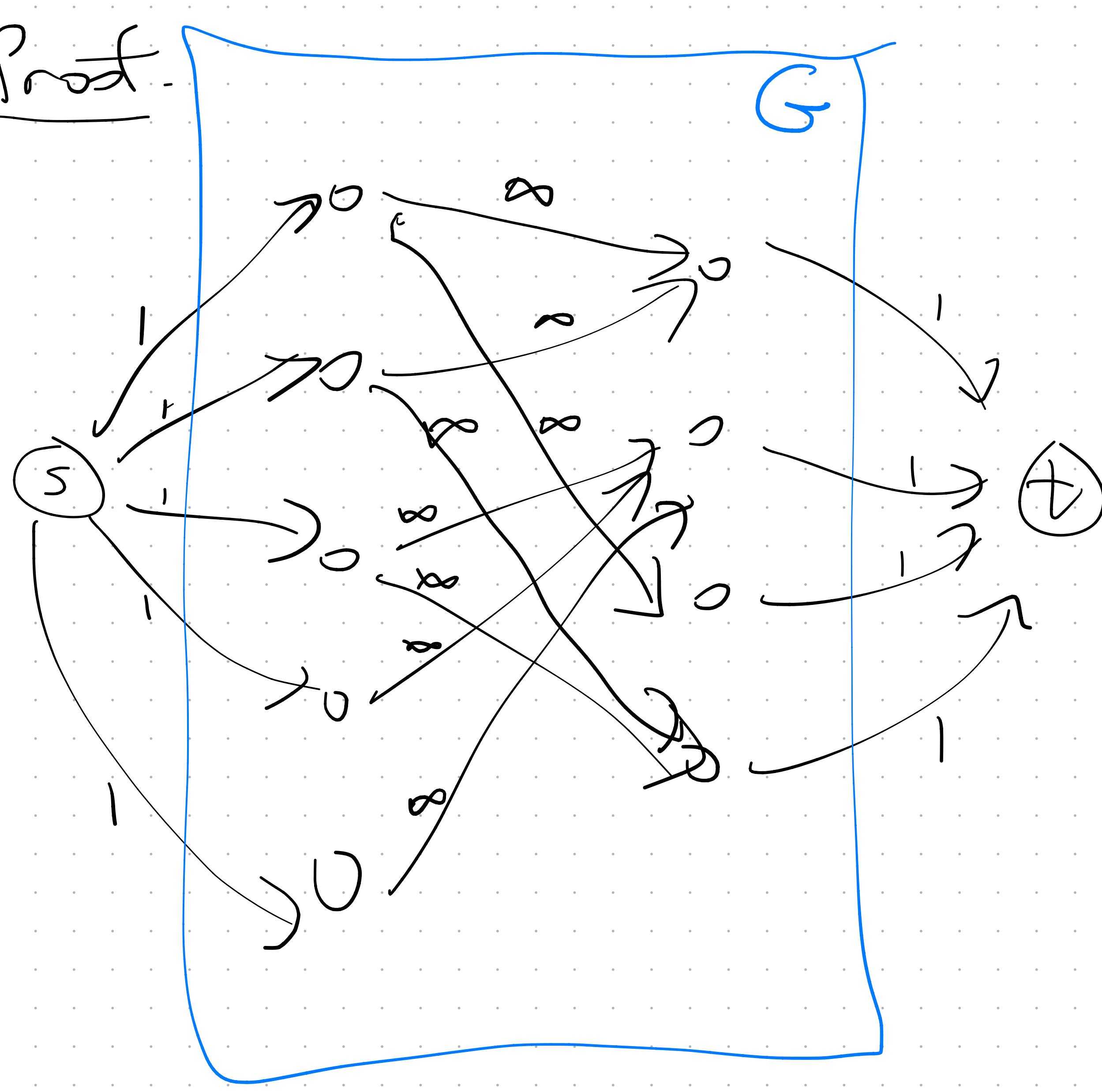


Def. A vertex cover of a graph G is a set of vertices, C, such that every edge has at least one endpoint in C.

König – Egervary Theorem.

In every (finite) bipartite graph the max # of edges in a matching equals the min # of vertices in a vertex cover.

Proof.



max matching

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max flow

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min cut

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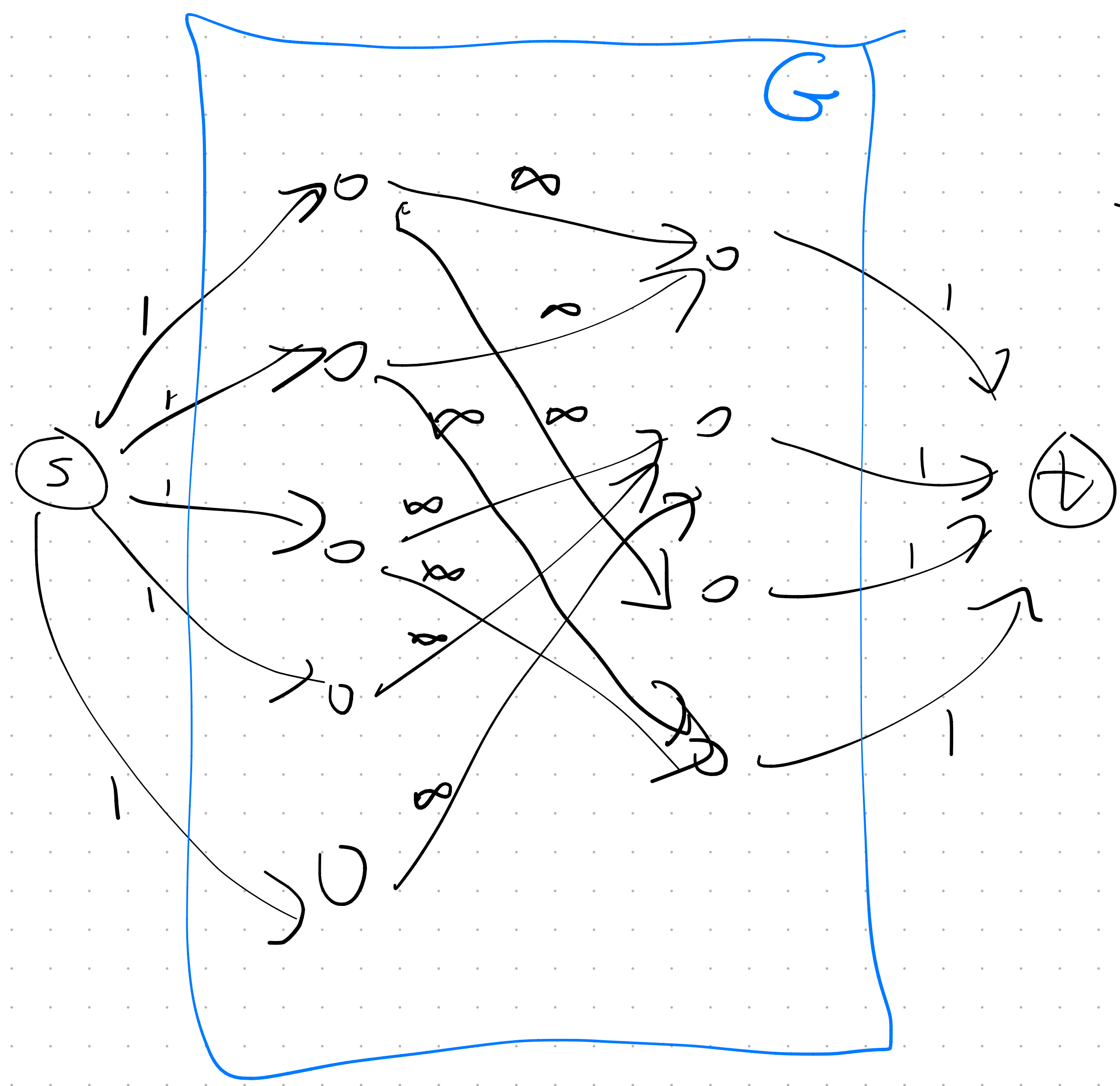
min vtx cover



max matching

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max flow

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min cut

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min vtx cover

Obs. If A, B is an s-t cut of finite capacity, then there are no ∞-capacity edges from A to B.

$A \cap L$ has no edges to $B \cap R$.

3 types of edges in middle layer:

$A \cap L \longrightarrow B \cap L \qquad A \cap R \longrightarrow B \cap L$

$A \cap R \longrightarrow B \cap R$

$$C = (B \cap L) \cup (A \cap R)$$

$$|C| = \mathrm{cap}(A, B)$$

$$\|$$

$$|\text{max matching}| = \mathrm{Val}(\text{max flow})$$