15 Oct 2025

Dinitz Algorithm

"Like Hopcroft–Karp for max-flow."

Def. In a flow network (G, c, s, t) an edge is advancing if it belongs to a shortest s-t path.

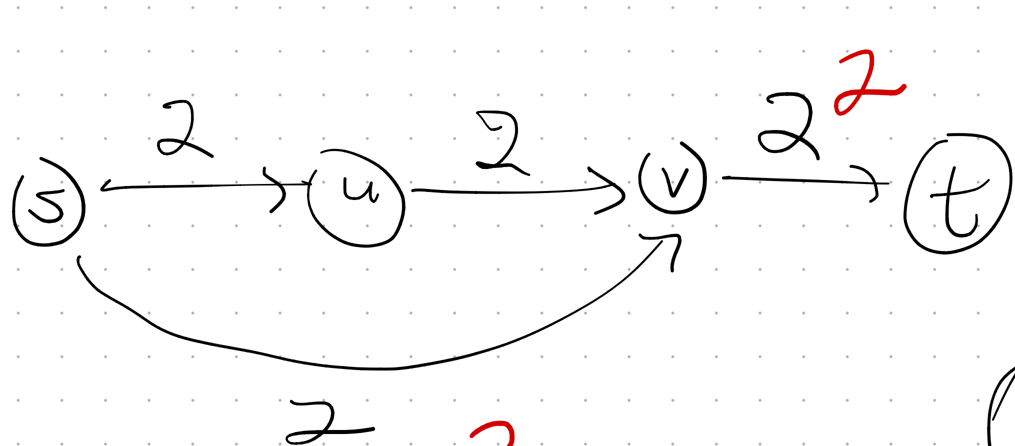
A blocking flow in G is a feasible flow that uses only advancing edges **and** saturates at least one edge of every shortest s-t path in G. (Equivalently, every s-t path composed entirely of advancing edges.)

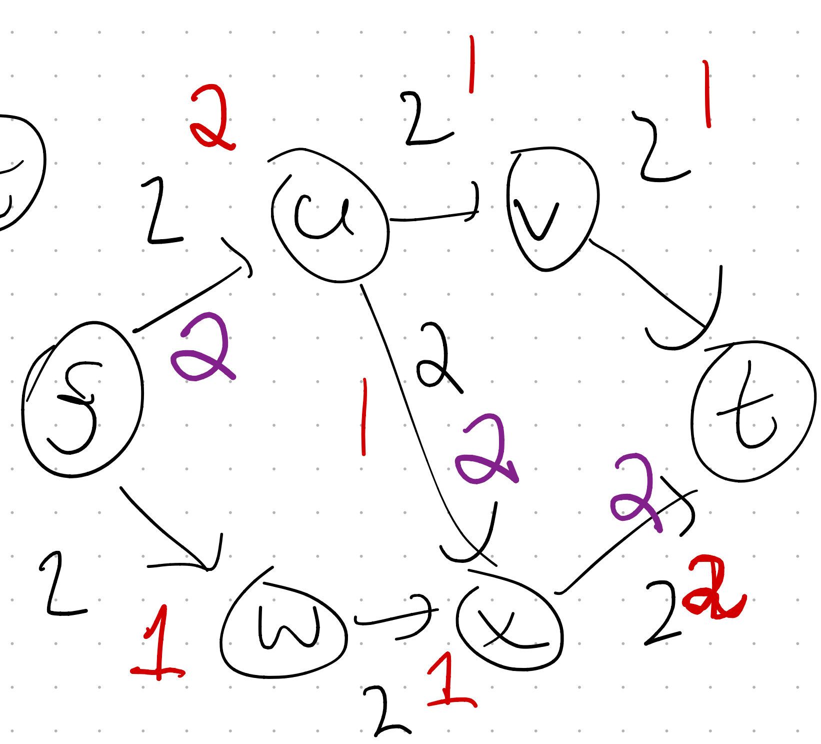
If $h(u,v) > 0$ then (u,v) is advancing

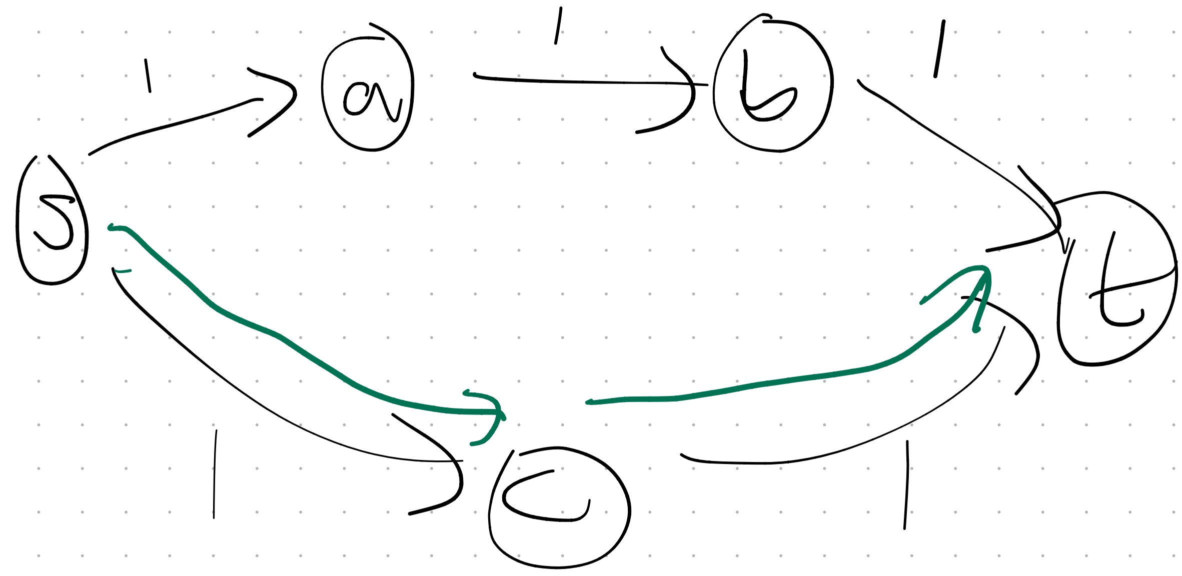
Saturate means $h(u,v) = c(u,v)$

If $d(u)$ denotes # of edges in a shortest path from s to u then every advancing edge (u,v) must satisfy

$$d(v) = d(u) + 1.$$







DINITZ ALGORITHM (outer loop)

$f = 0$. $G^f = (V, E^f)$

while G^f contains an s-t path

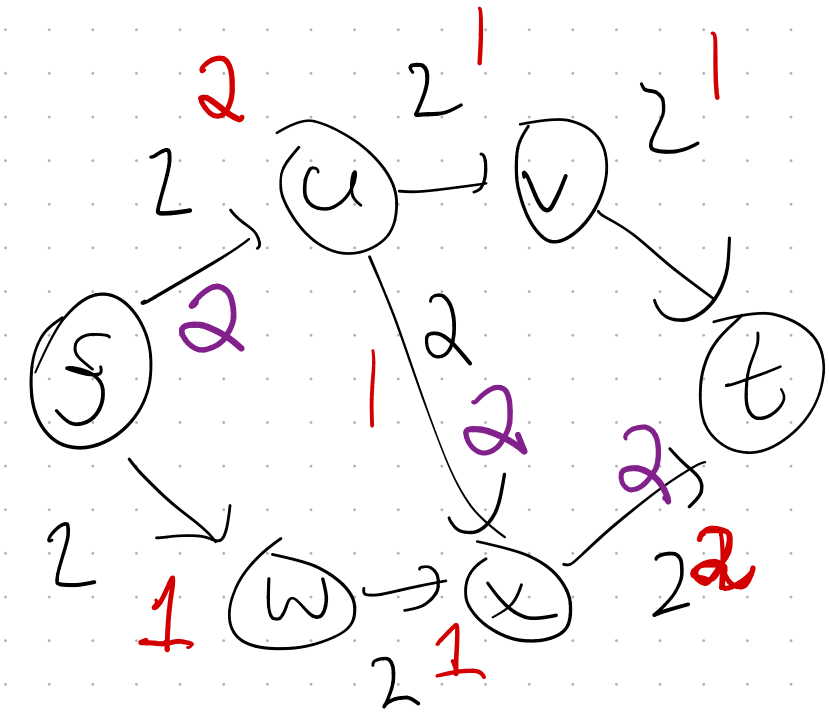
  Let h be a blocking flow in G^f.

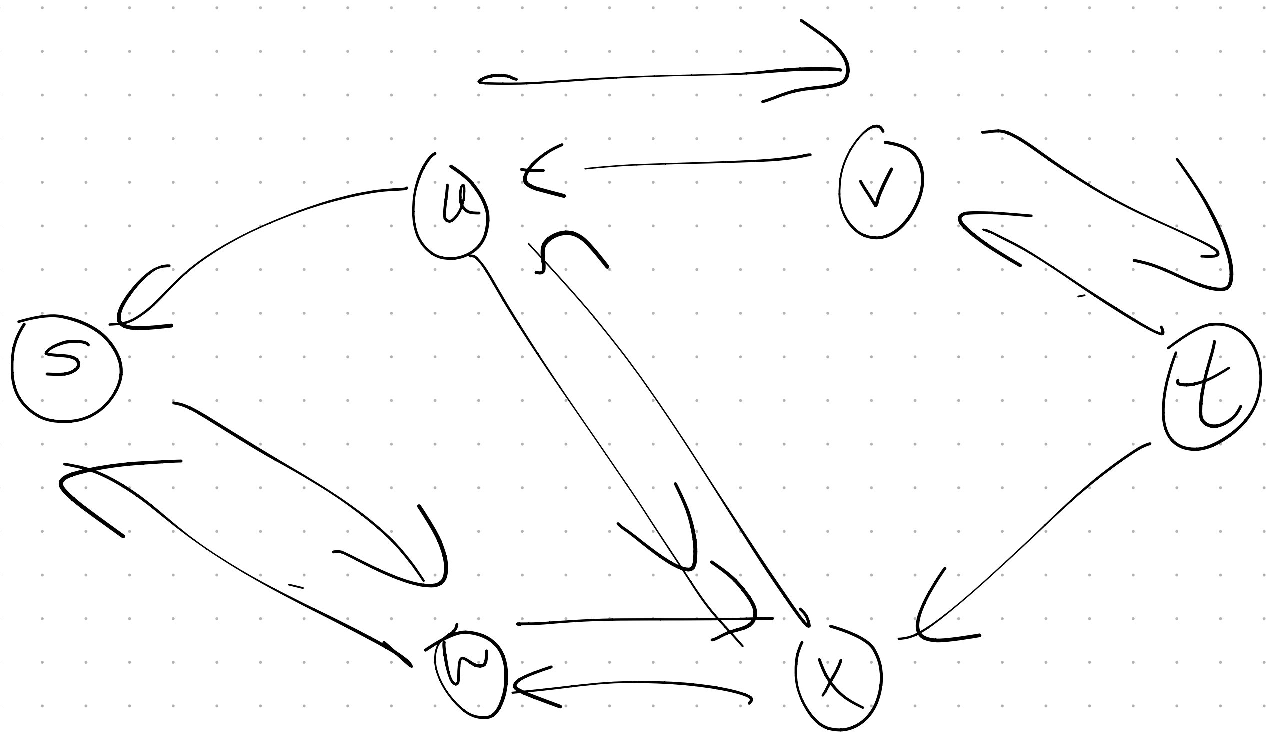
  Update f to $f + h$.

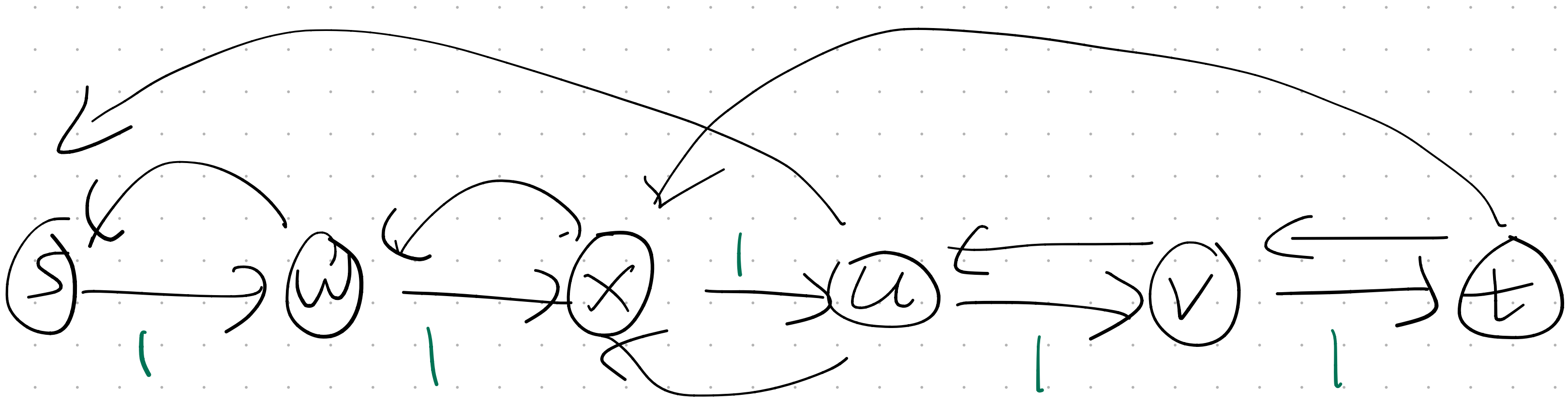
endwhile

output f

1. Bound # of outer loop iterations.
2. Talk about fast alg for inner loop.







<u>Lemma</u>: If h is a blocking flow in G^f, then the shortest s-t path in G^{f+h} is strictly longer than the shortest s-t path in G^f.

Recall, $d(u)$ = length of shortest s-u path.

Let $d^f(\cdot)$, $d^{fth}(\cdot)$ denote dist labels in G^f and G^{fth}.

Every edge of G^{fth}, say (u,v), is either:

- present in G^f too...

  then $d^f(v) \leq d^f(u) + 1$

or

- absent in G^f but $h(v,u) > 0$.

  then $d^f(u) = d^f(v) + 1$.

  $d^f(v) = d^f(u) - 1$.