10 Oct 2025

Def. For network G and flow f,

$$E^f = \{ (u,v) \mid f_{uv} < c_{uv} \}$$

$$= \{ (u,v) \mid c^f_{uv} > 0 \}.$$

An f-augmenting path is P from s to t with $E(P) \subseteq E^f$.

Def. An s-t cut is a partition of V into sets A, B with $s \in A$, $t \in B$.

Fact. If f is a flow, (A,B) is st cut then

$$\sum_{u \in A} \sum_{v \in B} f_{uv} = \mathrm{val}(f).$$

$$LHS + \underbrace{\emptyset}_{\text{skew symm}} = \sum_{u \in A} \sum_{v \in B} f_{uv} + \sum_{u \in A} \sum_{v \in A} f_{uv}$$

$$= \sum_{u \in A} \sum_{v \in V} f_{uv} \underset{\text{flow cons.}}{=} \sum_{v \in V} f_{sv} = RHS$$

Cor. If f is a feasible flow, (A,B) is st cut then

$$\mathrm{val}(f) \leq \sum_{u \in A} \sum_{v \in B} c_{uv} = c(A,B)$$

$$\mathrm{val}(\text{any feasible flow}) \leq c(\text{any st cut})$$

$$\therefore \max_{f \text{ feasible}} \{ \mathrm{val}(f) \} \leq \min_{\text{st cut } A,B} \{ c(A,B) \}.$$

Max-Flow Min-Cut Theorem:

$$\max_{f \text{ feasible}} \{ \text{val}(f) \} = \min_{s\text{-}t \text{ cut } A,B} \{ c(A,B) \}.$$

Proof. Recall if f is a feasible flow of max value then E^f contains no s-t path.

If $A = \{ u \mid E^f \text{ contains path } s \rightsquigarrow u \}$

$B = V - A$

then A, B is a s-t cut.

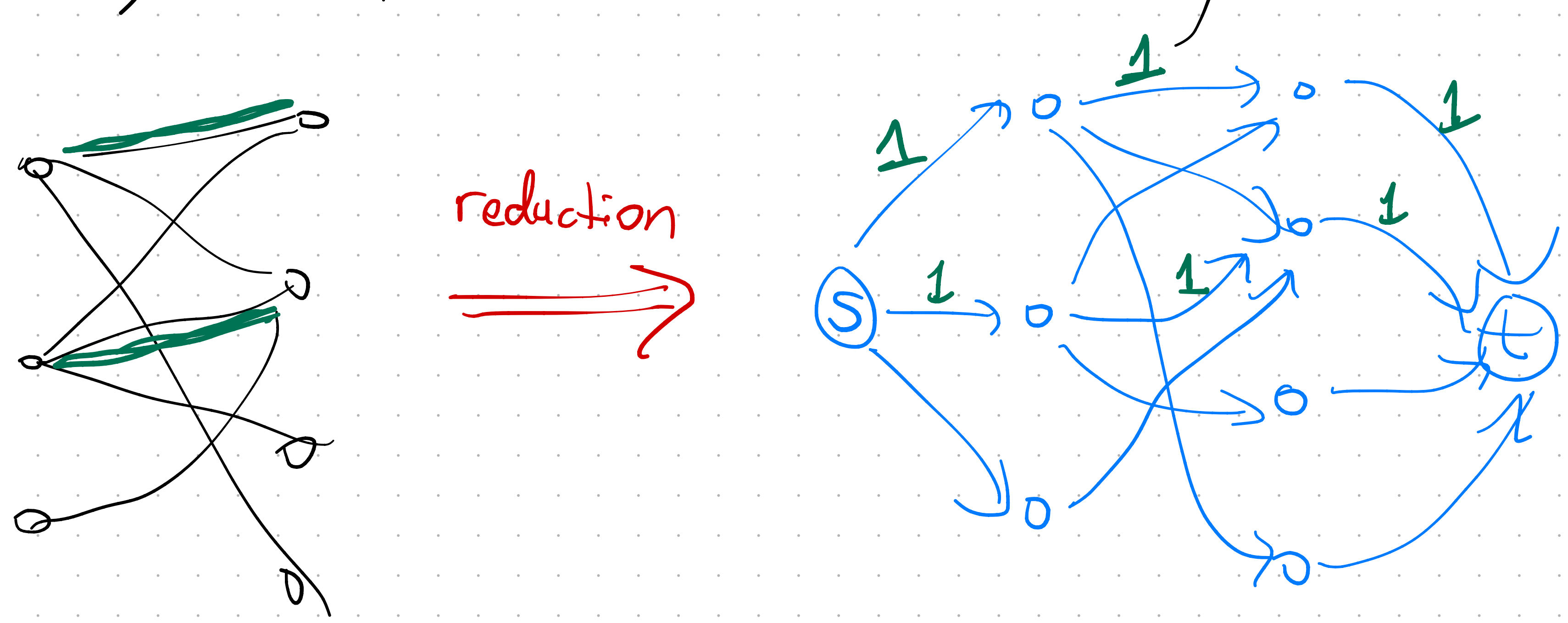
And $c^f_{uv} = 0 \quad \forall u \in A, v \in B$.

$\therefore \quad f_{uv} = c_{uv} \quad \forall u \in A, v \in B$

$$\therefore \quad \sum_{u \in A} \sum_{v \in B} f_{uv} = \sum_{u \in A} \sum_{v \in B} c_{uv} = c(A,B)$$

$= \text{val}(f)$.

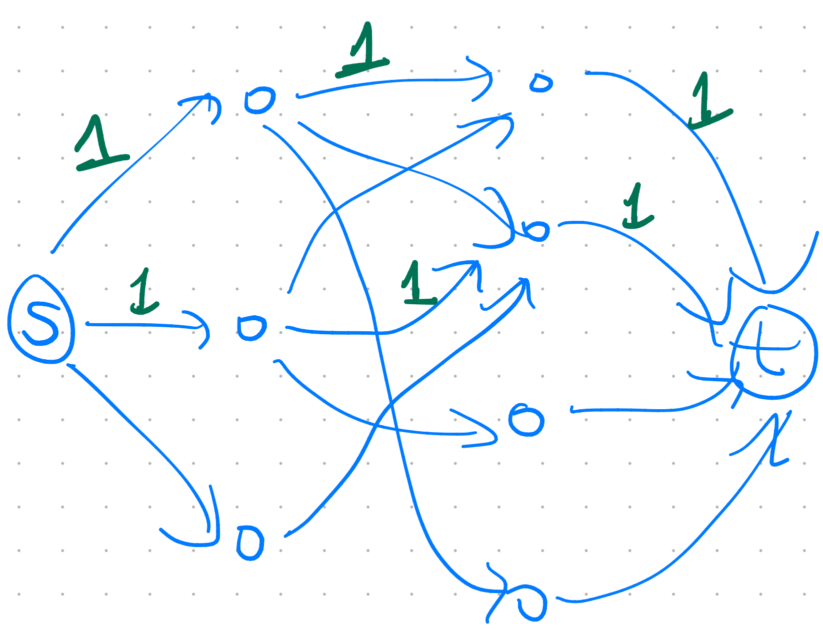
Relating bipartite max cardinality matching to flow.



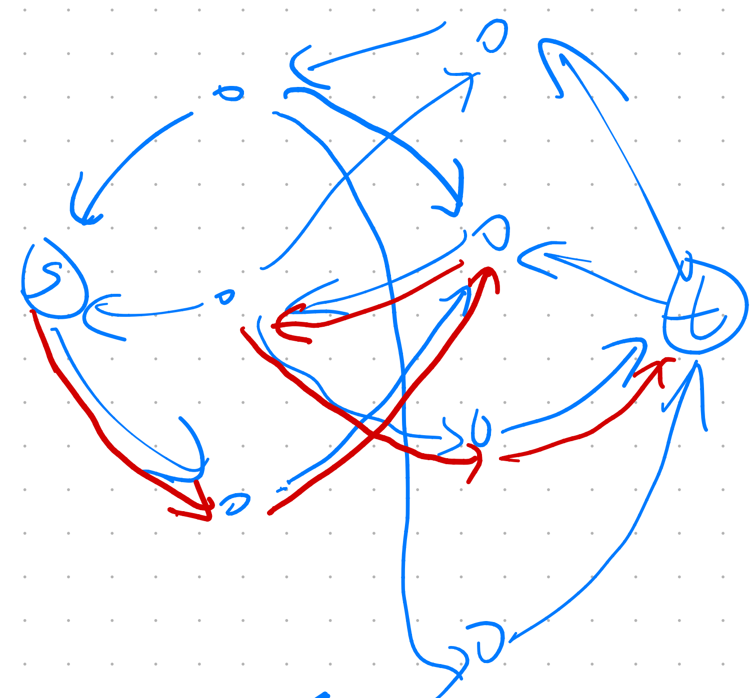
All edges capacity 1.

$\{\text{matchings}\} \longleftrightarrow \{\text{feasible integer flows}\}$

(cardinality) $\rightsquigarrow$ (value of flow)



All edges capacity 1.

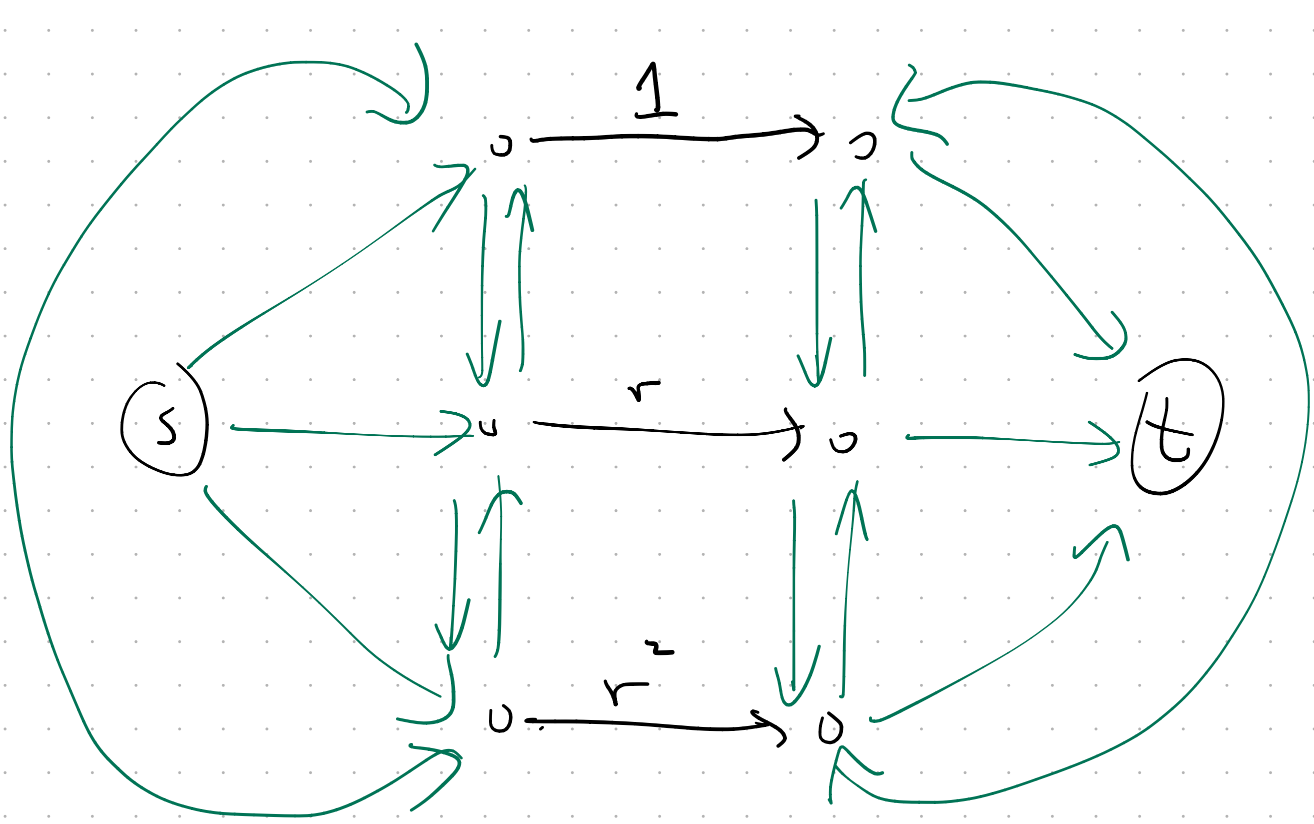


$\Uparrow$

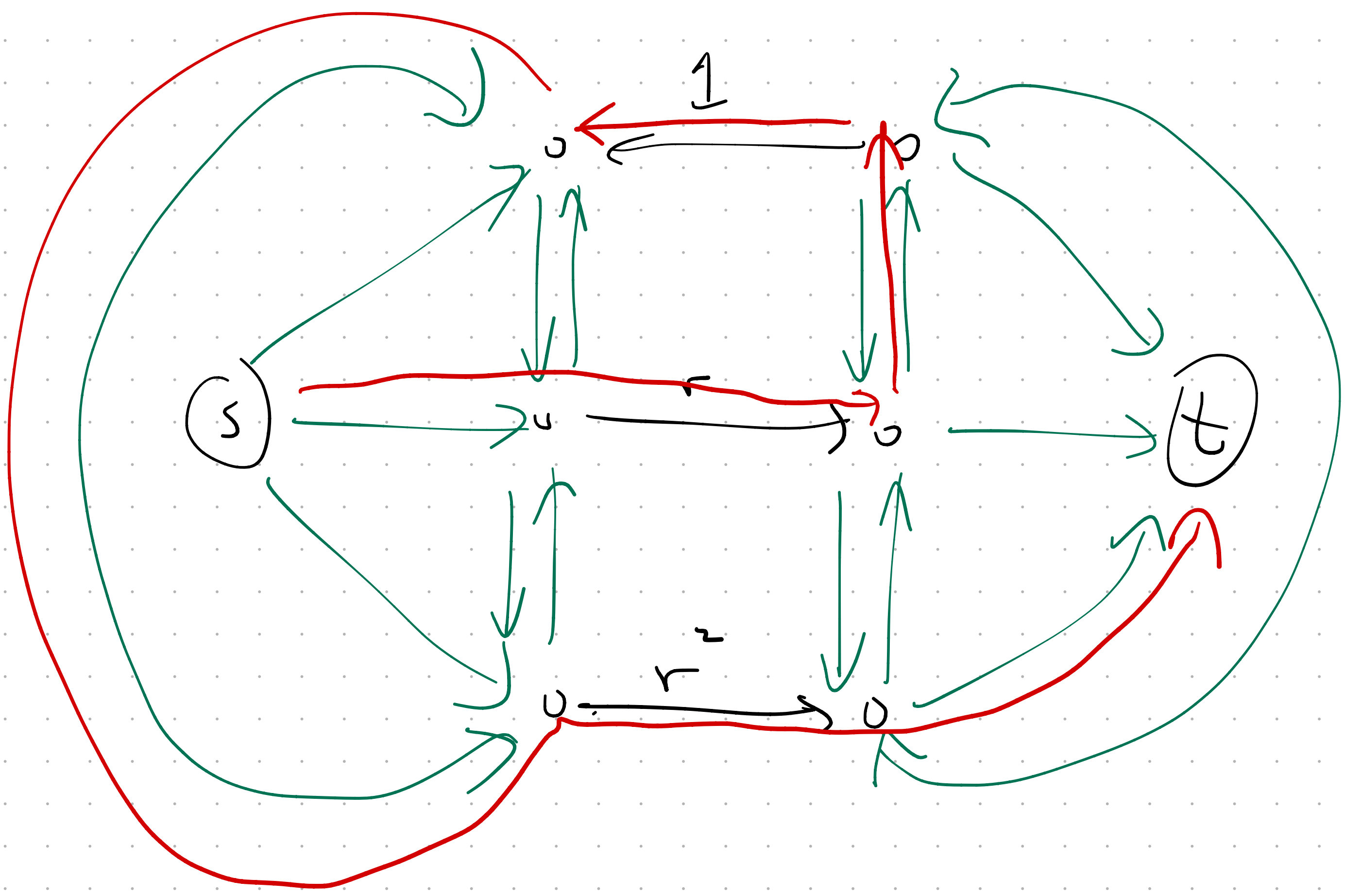
G_M

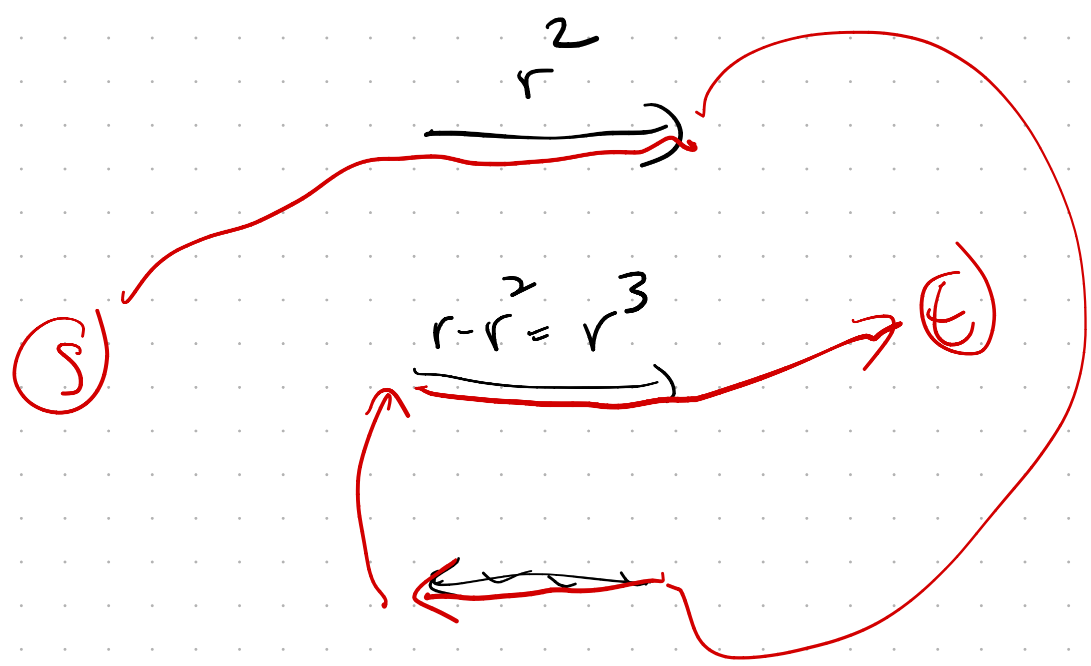
Let r satisfy $r + r^2 = 1$.

$$r = \frac{\sqrt{5} - 1}{2} = 0.618\ldots$$



Green edges have capacity 100.


1
s
r
t
r^2


r^2
s
$r - r^2 = r^3$
t