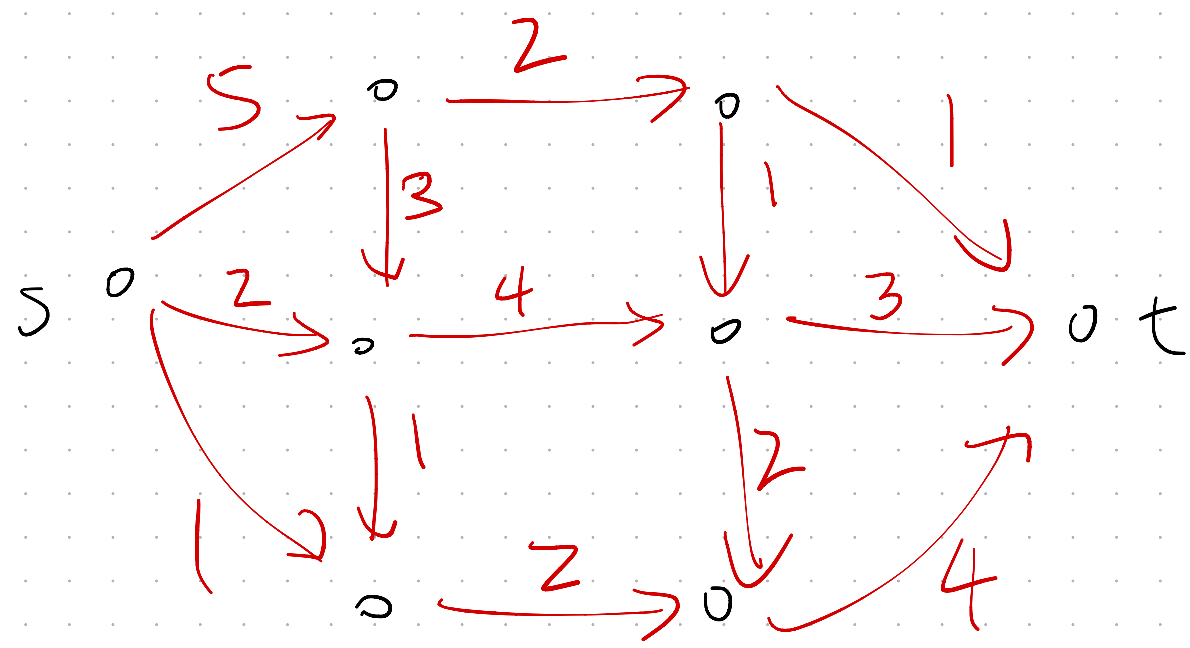
8 Oct 2025 — Max Flow



For network G with capacities c and flow f, the residual network G^f has capacities

$$c^f_{uv} = c_{uv} - f_{uv}$$

Observation: For any network G with feasible flow f, there is a bijection

$$\{\text{feasible flows in } G\} \longleftrightarrow \{\text{feasible flows in } G^f\}$$

$$f + f' \longleftarrow f'$$

$$f'' \longmapsto f'' - f$$

$\forall uv \quad f'_{uv} \leq c^f_{uv}$

$f'_{uv} + f_{uv} \leq c_{uv}$

Def. The value of f, $val(f)$, is

$$val(f) = \sum_{u \in V} f_{su}$$

$$= \sum_{u \in V} f_{ut}$$

A maximum flow is a feasible f that maximizes $val(f)$.

f $\qquad\qquad$ G^f

$s \circ \xrightarrow{1} \circ\, t \qquad \circ \xrightarrow{1} \circ \qquad s\, \circ \xleftarrow{1} \circ\, t$

$f'' \qquad\qquad f'' - f$

$\circ \xrightarrow{1/2} \circ \qquad \circ \xleftarrow{1/2} \circ$

$$val(f') + val(f) \longleftarrow\!\shortmid\; val(f')$$

$$f + f' \longleftarrow\!\shortmid\; f'$$

$$f'' \longmapsto f'' - f$$

$$val(f'') \longmapsto val(f'') - val(f)$$

Finding a flow of posi value in G^f means finding a way to modify f to a flow of greater value in G.

__Lemma.__ The following are equivalent:

a. f is a maximum flow in G. (a feasible flow of max value)

b. The maximum flow in G^f is zero.

c. In G^f there is no path from s to t made up of edges (u,v) with $c^f_{uv} > 0$.

__Proof.__ (a) $\Longleftrightarrow$ (b) by bijection.

(b) $\Leftrightarrow$ (c) If path P in G^f is composed of edges with capacity $c^f_{uv} \geq \delta$, and P

goes from s to t, then

let

$$f'_{uv} = \begin{cases} \delta & \text{if } (u,v) \in E(P) \\ -\delta & \text{if } (v,u) \in E(P) \\ 0 & \text{otherwise} \end{cases}$$

This f' is a flow of value δ

in G^f.

Conversely, if G^f has no

path from s to t composed

of edges with $c^f_{uv} > 0$,

let

$$A = \left\{ v \;\middle|\; \begin{array}{l} G^f \text{ contains a path from} \\ s \text{ to } v \text{ made up} \\ \text{of edges with } c^f_{uv} > 0 \end{array} \right\}$$

$$B = V - A.$$

Note $s \in A$, $t \in B$.

For any feasible f' in G^f,

$$\mathrm{val}(f') = \sum_{v} f'_{sv}$$

$$= \sum_{u \in A} \sum_{v} f'_{uv}$$

$$= \underbrace{\sum_{u \in A} \sum_{v \in A} f'_{uv}}_{=0} + \underbrace{\sum_{u \in A} \sum_{v \in B} f'_{uv}}_{\leq 0}$$

$$\leq \sum_{u \in A} \sum_{v \in B} c^f_{uv}$$

$$= 0$$

FORD-FULKERSON ALGORITHM.

Initialize $f = 0$

while G^f has a path P
composed of edges with $c^f_{uv} > 0$

let $\delta = \min\{c^f_{uv} : (u,v) \in E(P)\}$

let $f'_{uv} = \begin{cases} +\delta & \text{for } u,v \in E(P) \\ -\delta & \text{for } v,u \in E(P) \\ 0 & \text{o.w} \end{cases}$

update $f \leftarrow f + f'$ // val improves by δ

endwhile

output f.

Note. If $c_{uv} \in \mathbb{N}$ for every (u,v), then f_{uv} and c^f_{uv} will be integer-valued in every iteration.

Cor. 1. For networks with $\mathbb{N}$-valued capacities, Ford-Fulkerson always terminates.

Cor. 2. With $\mathbb{N}$-valued capacities there always exists a max flow that is $\mathbb{N}$-valued.
("Flow integrality theorem")