$\boxed{\text{6 Oct 2025}}$ Max-Flow Min-Cut

Announcement: Deadline extension PSet 2 by one week.

Office hour 12:30-1:30 Tues (tomorrow).

<u>Def.</u> A <u>flow network</u> is $\quad n = |V| \quad m = |E|, \quad m \geq n$

- a directed graph $G = (V, E)$
- vertices s, t called "source" and "sink"
- capacities $c_{uv} \in [0, \infty]$
  where $c_{uv} = 0$ if $(u,v) \notin E$.

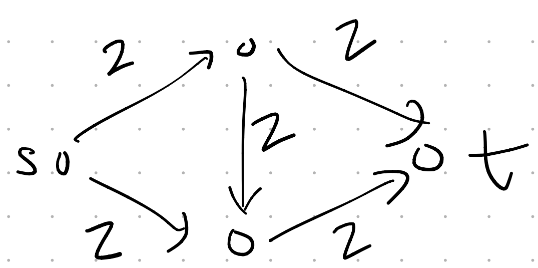
(Input size $O(m)$ real numbers.)

A <u>flow</u> is a function f assigning a number $f_{uv} \in \mathbb{R}$ to every pair $(u,v) \in V^2$, satisfying

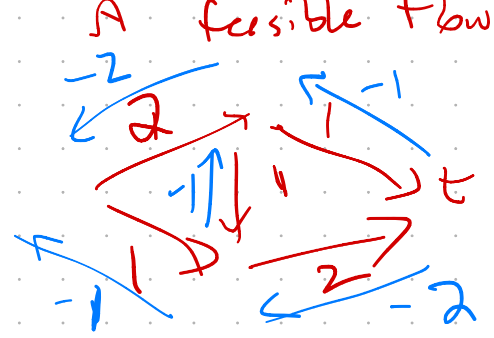
- [skew symmetry] $\quad f_{uv} + f_{vu} = 0 \quad \forall u, v$
- [flow conservation] $\quad \sum_{v \in V} f_{uv} = 0 \quad \forall u \notin \{s, t\}$.

A flow is <u>feasible</u> iff

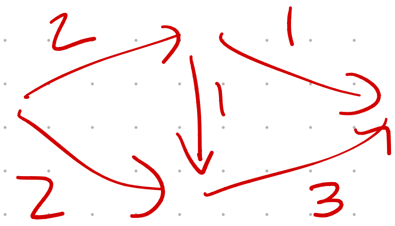
- [capacity constraints] $\quad f_{uv} \leq c_{uv} \quad \forall u, v.$



A feasible flow

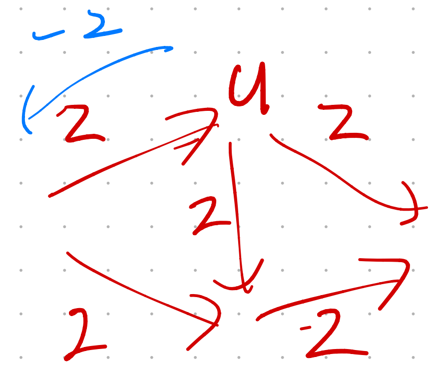


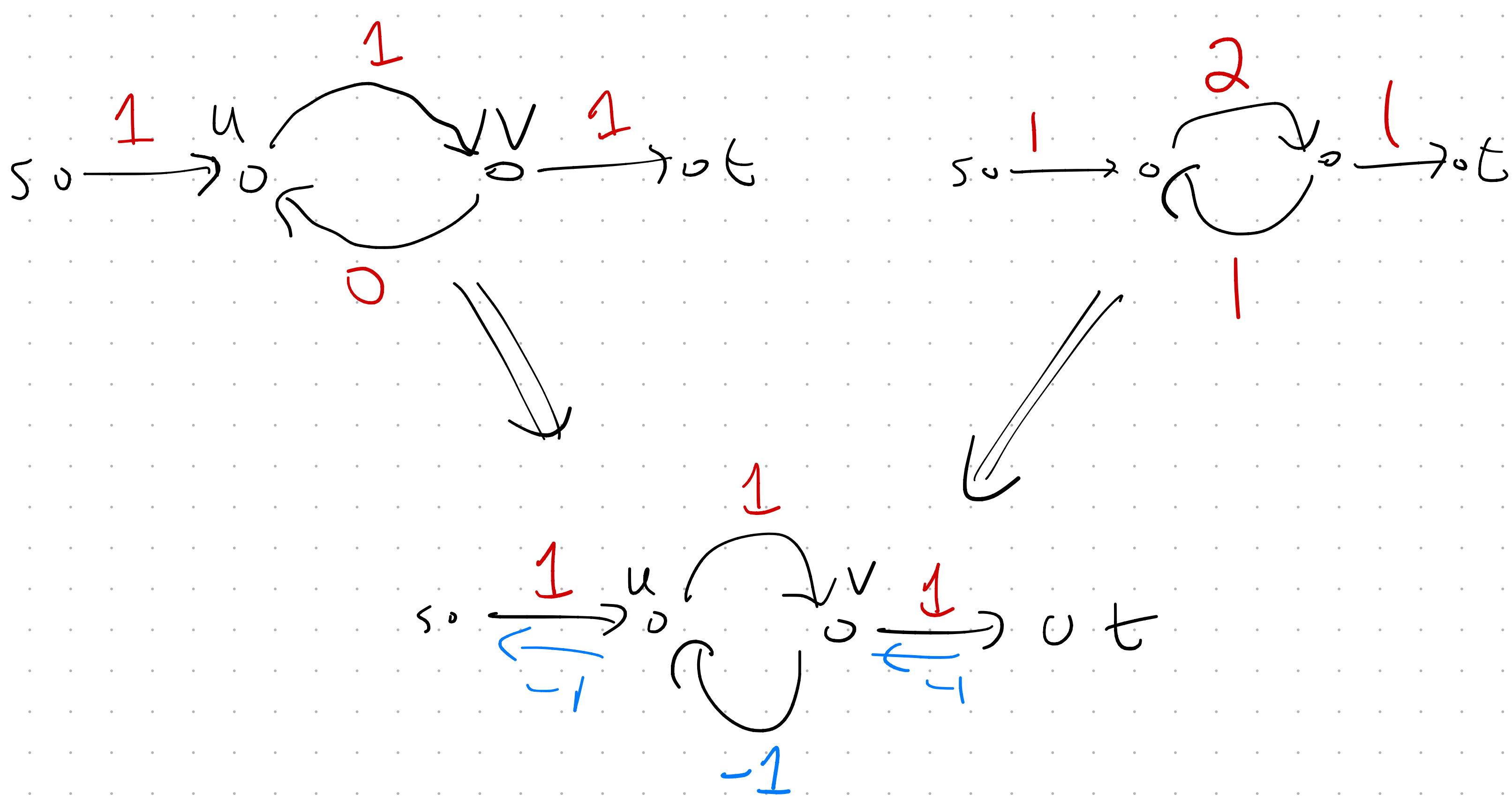
Flow, but not feasible



(Also negative vals on reverse edges.)

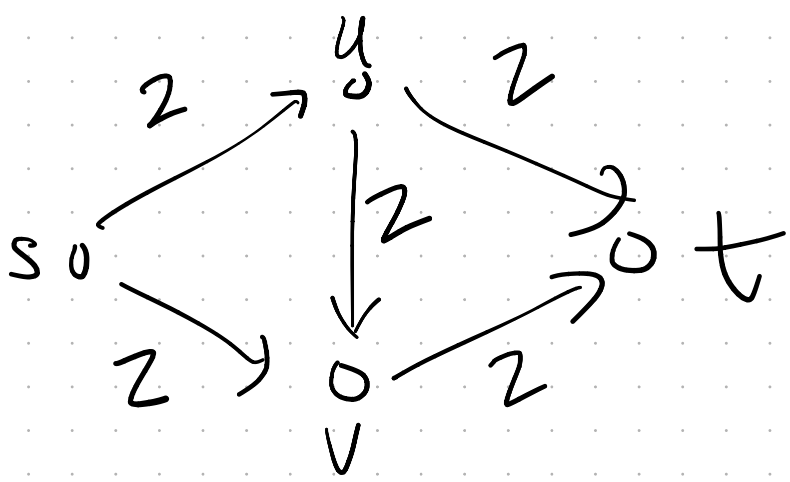
Feasible, but not flow

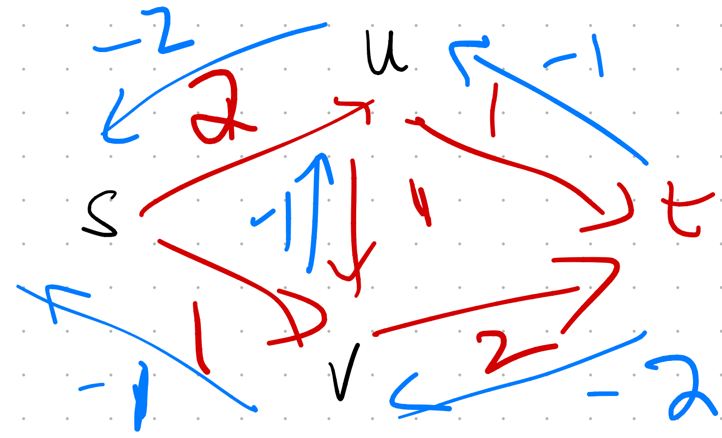




Def. For a flow network G and feasible flow f, the residual graph G^f is the flow network with vertex set V, source s, sink t, and capacities

$$c^f_{uv} = c_{uv} - f_{uv}.$$





$c_{uv} = 2$ $\qquad f_{uv} = 1$ $\qquad c^f_{uv} = 1$

$c_{vu} = 0$ $\qquad f_{vu} = -1$ $\qquad c^f_{vu} = 1$