$\boxed{\text{1 Oct 2025}}$ Algebraic Algorithms

Announcement. PSet 2 on Canvas.

6820 do $\{1,2\}$

5820 do $\{1,3\}$

Last thing about online matching.

Karp–Vazirani–Vazirani

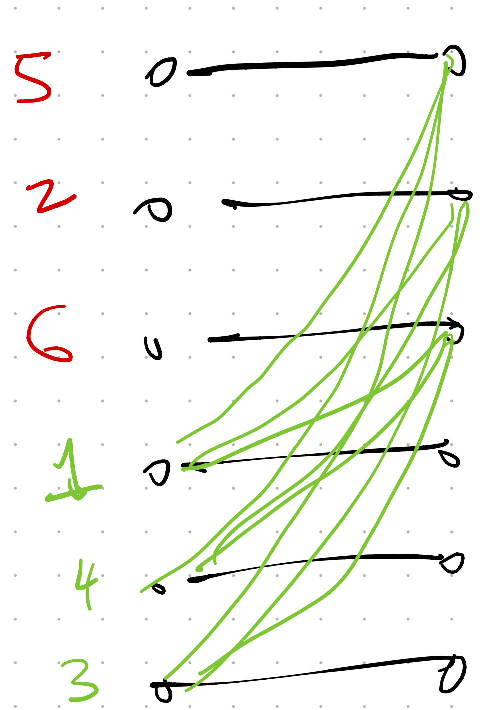
RANKING. randomized $(1-\frac{1}{e})$-competitive online matching algorithm that outputs genuine (i.e. non-fractional matching)

RANKING = "Greedy with Consistent-but-random tiebreaking"

At time $\emptyset$ sample a random permutation of L.

At time $t > 0$ when $v \in R$ arrives:

if v has a free neighbor match it to the one that occurs earliest in the rand permutation.



Using Determinants for Matchings (Lovász)

For bipartite graph with $|L| = |R| = n$ does it have a perfect matching?

Hopcroft–Karp solves in $O(m\sqrt{n}) \leq O(n^{2.5})$.

Lovász: randomized, solves in $O(n^{\omega})$.

$\omega =$ "exponent of matrix mult."

$= \inf \{ \alpha \mid \exists$ MatMult alg for $n \times n$ matrices using $O(n^{\alpha})$ algebraic ops $\}$

$$2 \leq \omega \leq 3,$$

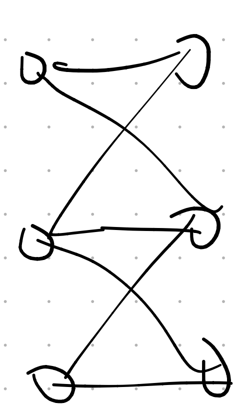
$$\omega \leq 2.372$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

← permutations on $[n]$

For bipartite G, form the $n \times n$ bipartite adjacency matrix A_G.

$$(A_G)_{ij} = \begin{cases} 1 & \text{if } (u_i, v_j) \in E(G) \\ \emptyset & \text{otherwise.} \end{cases}$$



$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{pmatrix} \#\text{ perf. matchings} \\ \text{in } G \end{pmatrix} = \underbrace{\sum_{\sigma \in S_n} \prod_{i=1}^{n} (A_G)_{i,\sigma(i)}}_{\text{permanent of } A_G.}$$