29 Sep 2025 — Online Matching: Waterfilling and Ranking

Announcement:

Problem Set 2 to be released tomorrow.

Write to me + Kevin using Ed if you need/want new partner(s).

(P)

$$\max \sum_{(u,v)\in E} x_{uv}$$

s.t. $\sum_{v\in R} x_{uv} \le 1 \quad \forall u\in L \quad (\alpha_u)$

$\sum_{u\in L} x_{uv} \le 1 \quad \forall v\in R \quad (\beta_v)$

$x_{uv} \ge 0 \quad \forall u,v$

(D)

$$\min \sum_{u\in L}\alpha_u + \sum_{v\in R}\beta_v$$

s.t. $\alpha_u + \beta_v \ge 1 \quad \forall (u,v)\in E$

$\alpha_u, \beta_v \ge 0$

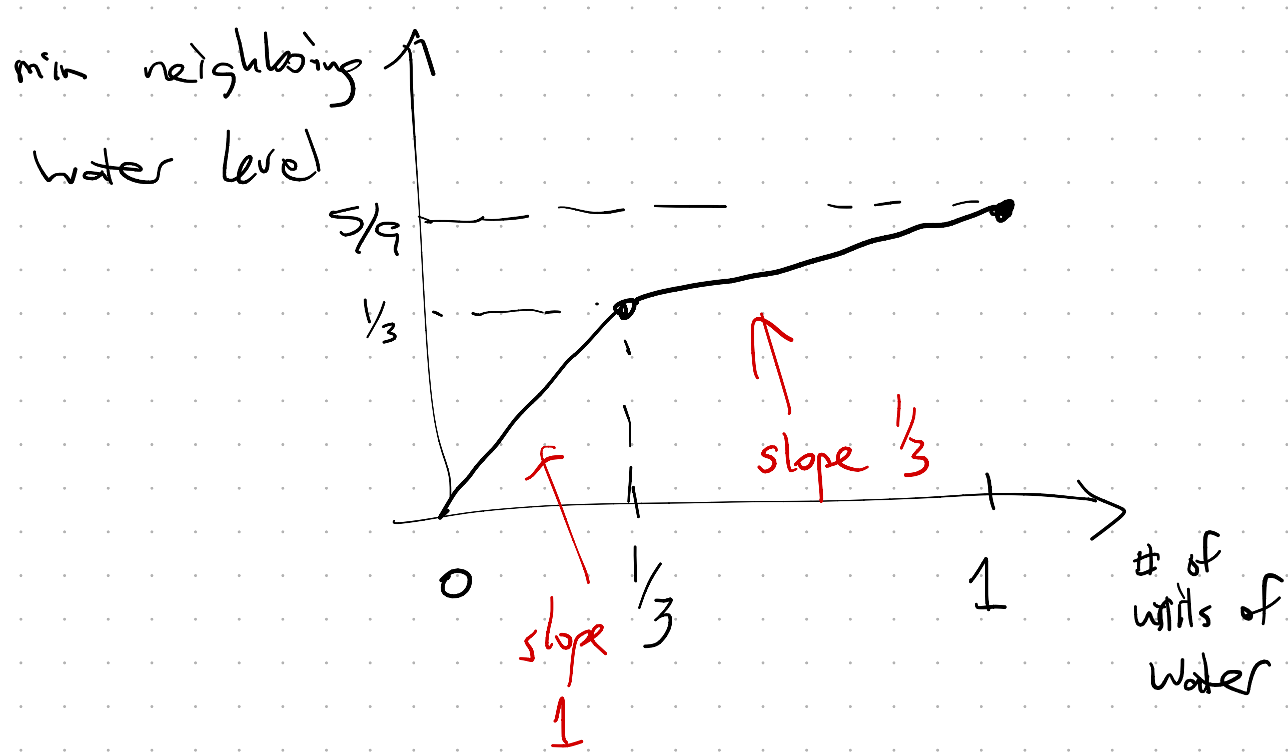
WATERFILLING: Keep track of a "water level" l_u for each left node, u.

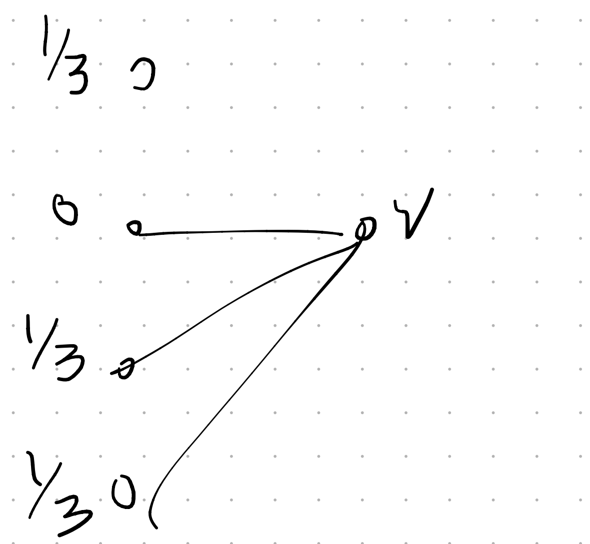
Each time a new vertex $v\in R$ arrives, it "pours" one unit of water into its neighbors by "continuously" filling the neighbors with lowest water level, until

(a) v has contributed 1 unit of water to the network

(b) every neighbor of v has reached water level 1.

As v pours its water into the neighbors, how does the min neighboring water level change over time?





We'll be setting $\alpha_u = g(l_u)$, where g is increasing, satisfying

$$g(0) = 0, \qquad g(1) = 1.$$

$$1 - g(x) + g'(x) = c \qquad \forall x \in [0,1].$$

Solving for c: let $h(x) = 1 - g(x)$.

$$h(0) = 1, \qquad h(1) = 0,$$

$$h - h' = c$$

$$h' = h - c$$

$$\frac{h'}{h-c} = 1$$

$$\ln(h(x) - c) = x + b$$

$$h(x) = e^{x+b} + c$$

$$e^b + c = 1$$

$$e^{b+1} + c = 0$$

$$e^b = 1 - c$$

$$e^{b+1} = e(1-c)$$

$$c(1-e) + e = 0$$

$$c = +\frac{e}{e-1}$$

$$g(y) = \frac{e^y - 1}{e-1} \qquad g(0) = 0 \qquad g(1) = 1$$

$$g'(y) = \frac{e^y}{e-1}$$

$$1 - g + g' = \frac{e\cancel{-1}}{e-1} - \frac{\cancel{e^y}\cancel{-1}}{e-1} + \frac{\cancel{e^y}}{e-1}$$

$$= \frac{e}{e-1}$$

$$\alpha_u = g(l_u)$$

$$\beta_v = 1 - \min\{\alpha_u \mid u \in N(v)\}$$

$$= 1 - \min\{g(l_u) \mid u \in N(v)\}$$

$$= 1 - g\left(\min_{u \in N(v)} \{l_u\}\right).$$

By design, $\alpha_u, \beta_v \geq 0$ and

$$\alpha_u + \beta_v \geq 1 \quad \forall \text{ edge } (u,v).$$

At time t during water-filling process

$$\hat{l}_v \overset{\Delta}{=} \min_{u \in N(v)} \{l_u\}.$$

defined to be

If k neighbors were at min water level as t increased to $t + dt$,

$$\hat{l}_v \text{ increased to } \hat{l}_v + \frac{dt}{k}.$$

For every $u \in N(v)$ that was at min level,

α_u increased from $g(\hat{\ell}_v)$ to $g(\hat{\ell}_v + \frac{dt}{k})$

increased by $g'(\hat{\ell}_v) \cdot \frac{dt}{k}$

Combined $\Delta(\sum \alpha_u)$ at time t:

$$g'(\hat{\ell}_v) \leq g'(\underbrace{\hat{\ell}_v \text{ at } t=1}_{\hat{\ell}_v(1)}).$$

$$\beta_v = 1 - g(\hat{\ell}_v(1)).$$

$$\Delta\left(\sum \alpha_u\right) + \Delta\left(\sum \beta_v\right) \leq g'(\hat{\ell}_v(1)) + 1 - g(\hat{\ell}_v(1)).$$

$$= \frac{e}{e-1}.$$