24 Sep 2025

Online matching: Waterfilling and Ranking

Announcements:

1. My OH today start 4:30.
2. Prelim room assignments (all 9/26 2:30–3:20)

CT: Bloomberg 165

ITH: (A–L) Gates 310

Last name

(M–Z) CIS 350

LP Relaxation of Bipartite Matching

(P)

$$\max \sum_{(u,v)\in E} x_{uv}$$

s.t. $\sum_{v\in R} x_{uv} \le 1 \quad \forall u\in L$

$\sum_{u\in L} x_{uv} \le 1 \quad \forall v\in R$

$x_{uv} \ge 0 \quad \forall u,v$

(D)

$$\min \sum_{u\in L} \alpha_u + \sum_{v\in R} \beta_v$$

s.t. $\alpha_u + \beta_v \ge 1 \quad \forall (u,v)\in E$

$\alpha_u, \beta_v \ge 0$

Recall: Online Greedy algorithms are $\frac{1}{2}$-competitive, Idea: Augment Greedy alg to compute $X_{uv} = \begin{cases} 1 & \text{if } u \text{ matches to } v \\ \emptyset & \text{if not} \end{cases}$

and α_u, β_v values satisfying the (D) constraints.

Invariants:

1. At all times, $\vec{x}$ feasible for (P)
   $\vec{\alpha}, \vec{\beta}$ feasible for (D).

2. $$\sum_{(u,v)\in E} x_{uv} \geq \frac{1}{2}\left(\sum_{u\in L}\alpha_u + \sum_{v\in R}\beta_v\right).$$

$$\|\!\!\|$$

Max matching size

[weak duality] $\|\!\!\|$

$$\sum_{u\in L}\alpha_u + \sum_{v\in R}\beta_v$$

For Online Greedy, when it matches v to u, set

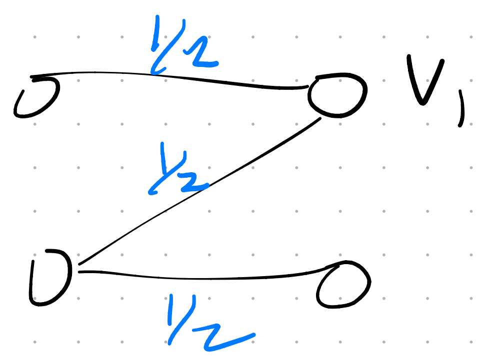
$$X_{uv} = 1$$

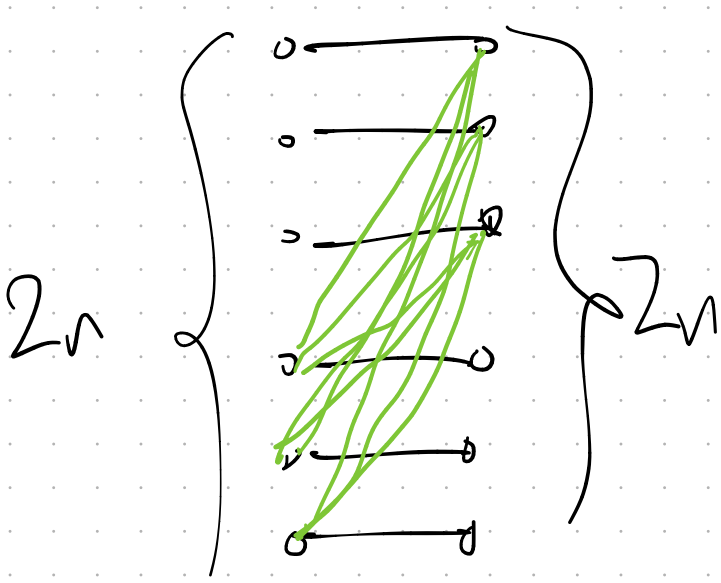
$$\alpha_u = \beta_v = 1.$$

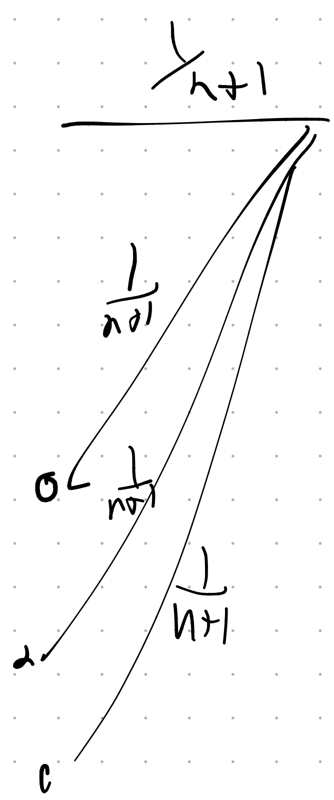
All var's initialized to $\emptyset$, remain $\emptyset$

unless set to 1 according to formulas above.

Online Fractional Matching.







After first n arrivals on R, here are the fractional degrees on L.

- $\frac{1}{n+1}$
- $\frac{1}{n+1}$
- $\frac{1}{n+1}$
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- $\frac{n}{n+1}$
- $\frac{n}{n+1}$

The remaining n nodes in R can only contribute $\frac{1}{n+1}$ each to the fractional matching.

Final matching size: $\frac{n}{n+1} + n < n+1$

Waterfilling: fractionally subdivides $(x_{uv})_{u \in L}$ among neighbors of v to maximize the minimum neighboring water level.

Water level $l(u) = \sum_{v \in R} x_{uv}$.