22 Sep 2025

Online Bipartite Matching

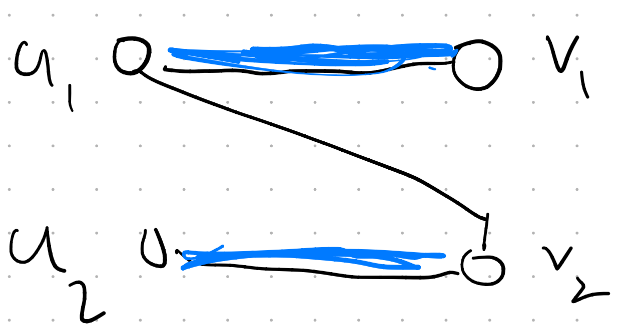
We have fixed node set L.

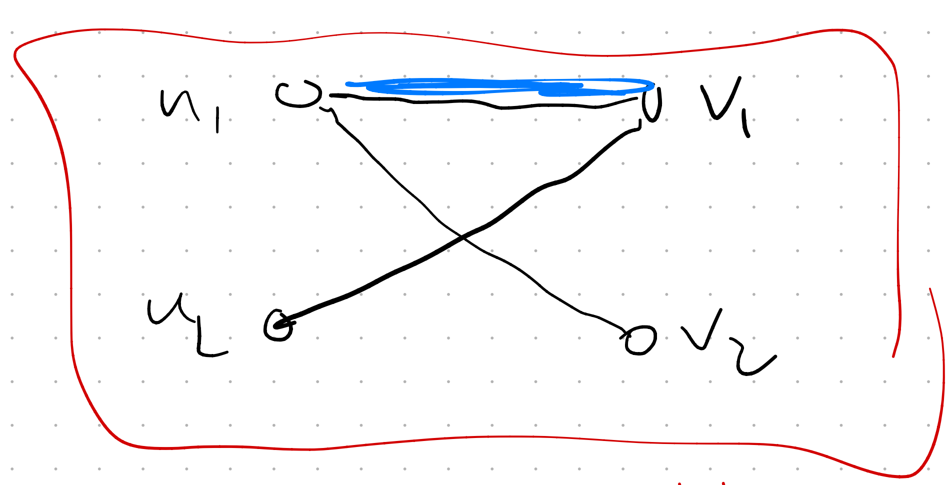
Requests (nodes) form set R, arrive one by one: $v_1, v_2, v_3, \ldots, v_n$.

When v_t arrives at time t (but not before then) its neighbor set becomes known.

An algorithm then chooses one (free) neighbor to match to v_t before v_{t+1} arrives. Or can choose to leave unmatched.

Goal: make as many matches as possible.





Illustrates you can't always compute a max matching online.

<u>Competitive ratio.</u> An algorithm that forms a matching with m_{ALG} edges when the max matching has m_{OPT} edges is called α-competitive if $\frac{m_{ALG}}{m_{OPT}} \geq \alpha$ for all input sequences.

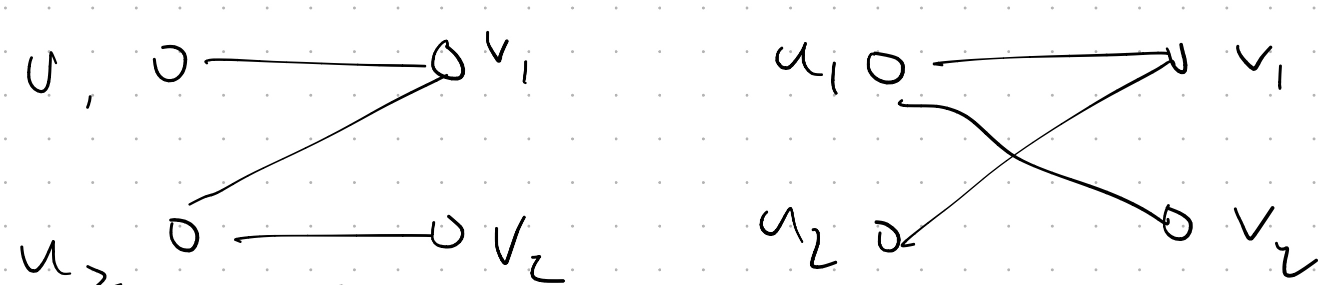
No (deterministic) algorithm is ~~better~~ than $\frac{1}{2}$-competitive for this problem.

For ~~randomized~~ algorithms, α-competitive means

$$\frac{\mathbb{E}[m_{ALG}]}{m_{OPT}} \geq \alpha \quad \forall \text{ input sequences,}$$

non-adaptive: graph structure and arrival order of R are predetermined before algo starts running.

Consider the following pair of inputs



I_1 $\qquad\qquad$ I_2

$m_{OPT}(I_1) = 2 \qquad\qquad m_{OPT}(I_2) = 2$

For any randomized algorithm

$$\mathbb{E}\left[m_{ALG}(I_1) + m_{ALG}(I_2)\right] \leq 3.$$

Why? Let $p_{11} = \Pr(\text{ALG matches } v_1 \text{ to } u_1)$.

$$p_{12} = \Pr(\ \text{—}\ v_1 \text{ to } u_2).$$

$$\mathbb{E}\, m_{ALG}(\sigma_1) \leq \underbrace{p_{11} + p_{12}}_{\Pr(v_1 \text{ matched})} + \underbrace{(1 - p_{12})}_{\Pr(v_2 \text{ matched})}$$

$$\mathbb{E}\, m_{ALG}(\sigma_2) \leq p_{11} + p_{12} + (1 - p_{11})$$

$$\mathbb{E}\left[m_{ALG}(\sigma_1) + m_{ALG}(\sigma_2) \right] \leq p_{11} + p_{12} + 2 \leq 3.$$

$$\frac{\mathbb{E}\, m_{ALG}(\sigma_1)}{m_{OPT}(\sigma_1)} + \frac{\mathbb{E}\, m_{ALG}(\sigma_2)}{m_{OPT}(\sigma_2)} \leq \frac{3}{2}$$

Either $\dfrac{\mathbb{E}\, m_{ALG}(\sigma_1)}{m_{OPT}(\sigma_1)} \leq \dfrac{3}{4}$ or $\dfrac{\mathbb{E}\, m_{ALG}(\sigma_2)}{m_{OPT}(\sigma_2)} \leq \dfrac{3}{4}$

No rand. alg. can be better than $\frac{3}{4}$-competitive.

<u>Def.</u> An online matching algorithm is <u>greedy</u> if it always matches an arriving node that has a free neighbor.

FACT. Greedy $\Longrightarrow$ $\frac{1}{2}$ - competitive.

Proof. This was (essentially) on the homework. You proved every maximal matching has $\geq \frac{1}{2}$ as many edges as a maximum matching.

[PRIMAL]

$$\max \sum_{(u,v)\in E} x_{uv}$$

$$\text{s.t.} \quad \sum_{v} x_{uv} \leq 1 \quad \forall u \in L$$

$$\sum_{u} x_{uv} \leq 1 \quad \forall v \in R$$

$$x_{uv} \geq 0$$

[DUAL]

$$\min \sum_{w \in L\cup R} y_w$$

$$\text{s.t.} \quad y_u + y_v \geq 1 \quad \forall (u,v) \in E$$

$$y_w \geq 0 \quad \forall w$$