$\boxed{\text{17 Sep 2025}}$ Finishing Duality + Simplex Alg

Announcement: Prelim covers through 9/19 lecture.

$$
\begin{array}{lll}
 & (P) & \\
\max & c^T x & \\
\text{s.t.} & Ax \preceq b & \\
 & x \succeq 0 &
\end{array}
=
\begin{array}{lll}
 & (D) & \\
\min & b^T y & \\
\text{s.t.} & A^T y \succeq c & \\
 & y \succeq 0 &
\end{array}
$$

Equivalent "equational form" of problem (P).

$$
\begin{array}{ll}
\max & c^T x \\
\text{s.t.} & Ax + w = b \\
 & x, w \succeq 0
\end{array}
\qquad x \in \mathbb{R}^n \quad w \in \mathbb{R}^m
$$

If the set of (x, w) where the max is attained is non-empty, it contains a point where n of the variables $\{x_1, \dots, x_n, w_1, \dots, w_m\}$ are zero.

Of the points x, w where max is attained $\exists$ one with as many 0 coord's as possible. (x^*, w^*)

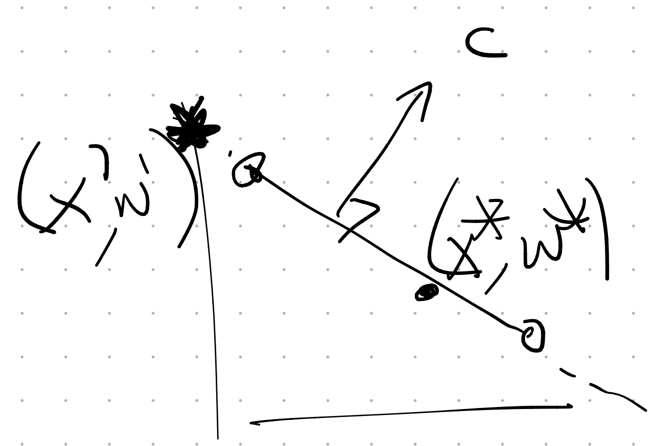
Suppose $n-1$ or fewer coord's are zero.
Augment $Ax + w = b$ with $n-1$ or fewer

equations of the form $x_i = 0$ or $w_j = 0$ corresponding to the coords that are zero at the ostensibly opt point.

The remaining coords of (x^*, w^*) strictly positive.

$\therefore$ (x^*, w^*) belongs to interior of a line segment in $\mathbb{R}^{m+n}_{\geq 0}$ satisfying all $m+n-1$ (or fewer) of these.

$c^T x$ must be const along this segment.



Extend segment to an "exit point" (x', w') where it exits the orthant $\mathbb{R}^n_{\geq 0}$.

$c^T x' = c^T x^*$ $\quad (x', w') \in \mathbb{R}^{n+m}_{\geq 0}$ $\quad Ax' + w' = b$

... and x' has at least one more coordinate equal to zero. ↯

If $S \subset [m+n]$ denotes the set of n coordinates of (x^*, w^*) that are 0,

then $\mathbb{1}_S = \begin{array}{c} \\ n \end{array} \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}^{m+n}$ e.g. $S = \{1,3,5,6\}$.

("coordinate projection matrix").

$Ax^* + w^* = b$

The S coord's of (x^*, w^*) equal zero

$$\left.\begin{array}{l} Ax^* + w^* = b \\ \text{The } S \text{ coord's of } (x^*, w^*) \text{ equal zero} \end{array}\right\} \quad \underbrace{\begin{bmatrix} A & \mathbb{1}^{m\times m} \\ \hline \multicolumn{2}{c}{\mathbb{1}_S} \end{bmatrix}}_{M_S} \begin{bmatrix} x^* \\ w^* \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

We can find S s.t. $M_S \begin{bmatrix} x^* \\ w^* \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$

and M_S is invertible.

Then $\begin{bmatrix} x^* \\ w^* \end{bmatrix} = M_S^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix}$.

(Note this reduces LP to searching over $S \subset [n+m]$.)

Every other feasible (x, w) is of the form $M_S^{-1} \begin{bmatrix} b \\ v \end{bmatrix}$ for some $v \geq 0$.

Reason: if (x,w) is feasible,

$$M_S \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Ax + w \\ \mathbb{1}_S \begin{bmatrix} x \\ w \end{bmatrix} \end{bmatrix} = \begin{bmatrix} b \\ \geq 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ w \end{bmatrix} = M_S^{-1} \begin{bmatrix} b \\ v \end{bmatrix} \text{ for } v \geq 0.$$

The objective function $c^T x$ can be expressed as

$$\begin{bmatrix} c^T & 0^T \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix} = \begin{bmatrix} c^T & 0^T \end{bmatrix} M_S^{-1} \begin{bmatrix} b \\ v \end{bmatrix}$$

As a function of v, the RHS is <u>affine</u> (constant plus linear).

At $v = 0$ it evaluates to $c^T x^* = OPT$.

The RHS can be written as

$$OPT - z^T x - y^T w.$$

This equals $c^T x$ for all x, w satisfying $Ax + w = b$.

$\forall x, w \quad Ax + w = b \Rightarrow c^T x = OPT - z^T x - y^T w$

$w = b - Ax$

$\forall x$

$$c^T x = OPT - z^T x - y^T (b - Ax)$$

$$= OPT - z^T x - y^T b + y^T A x$$

$$= (OPT - b^T y) + (y^T A - z^T) x$$

$\therefore$

$$b^T y = OPT$$

$$c = A^T y - z$$

$$A^T y = c + z.$$

<u>CLAIM.</u> $y \succeq 0$ && $z \succeq 0$.

This is implied by local optimality of (x^*, w^*).

(There's an issue with degeneracy ... see typeset notes.)