$\boxed{\text{15 Sep 2025}}$ Strong LP Duality

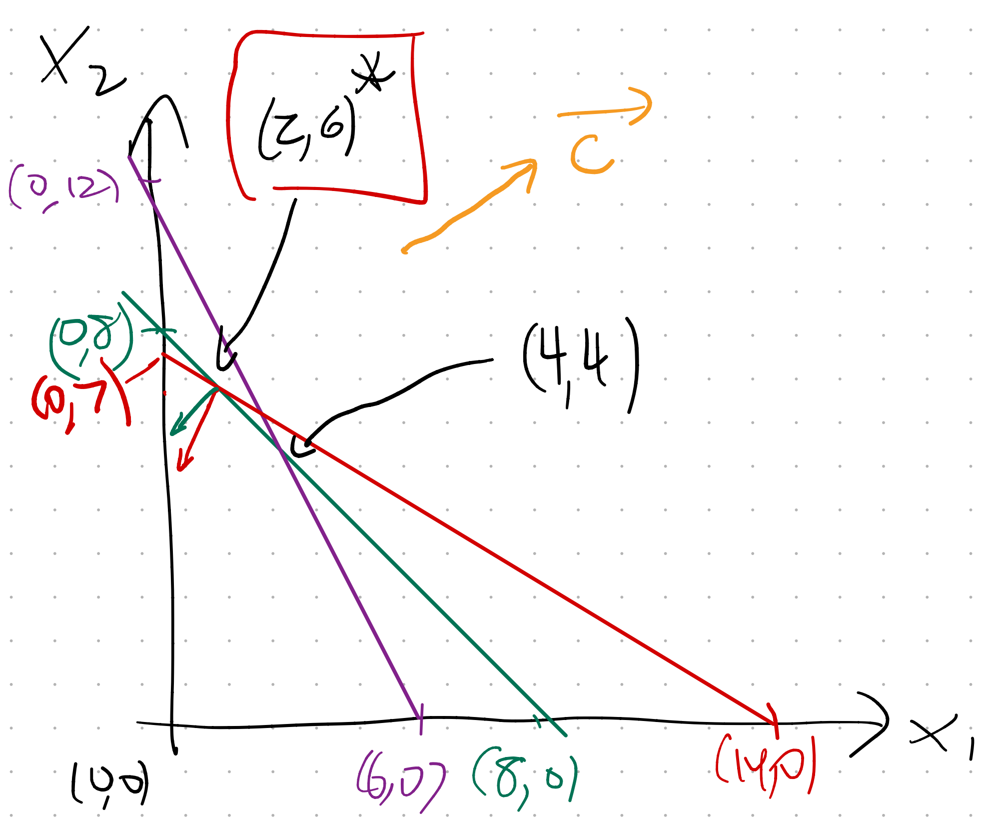
(Primal) $\underset{=\ (\text{strong})}{\overset{(\text{weak})}{\leq}}$ (Dual)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \succeq c \\ & y \succeq 0 \end{aligned}$$

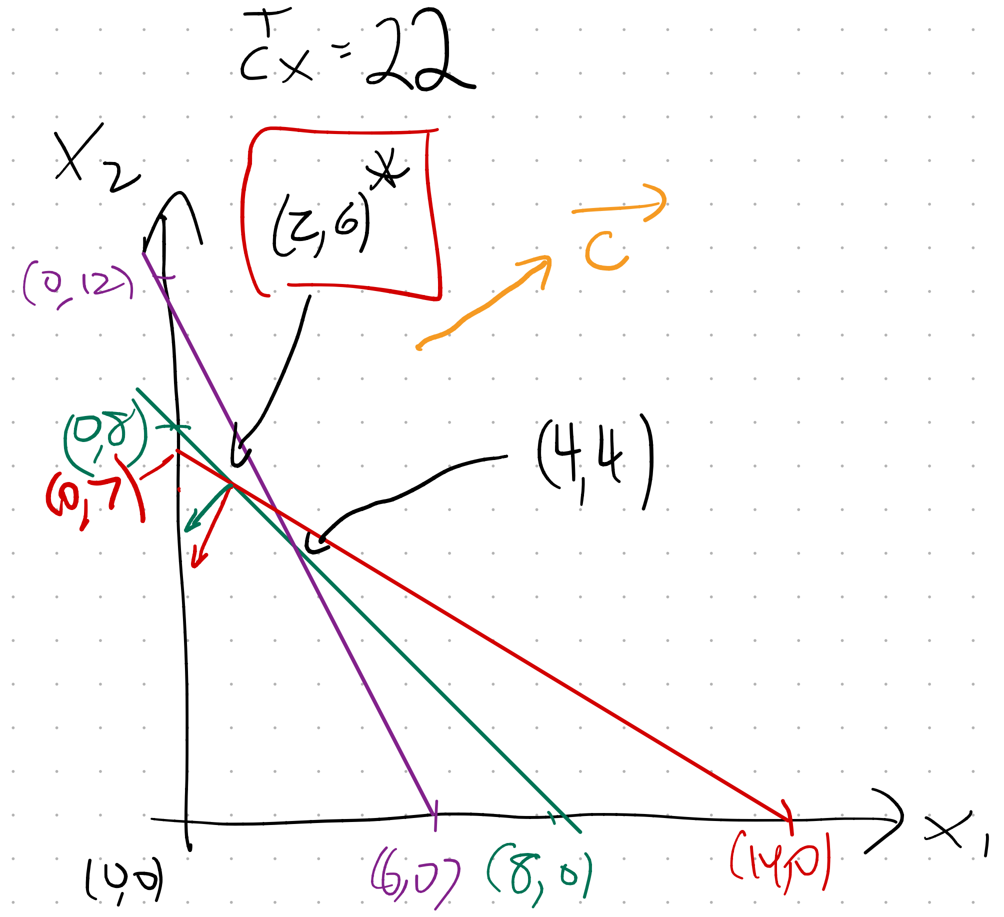
$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 + x_2 \leq 12 \\ & x_1 + 2x_2 \leq 14 \end{aligned}$$

$c^T x = 22 \qquad x_1, x_2 \geq 0$

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \preceq \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix} \\ & \vec{x} \succeq 0 \end{aligned}$$



$$\begin{aligned} \min \quad & \begin{bmatrix} 8 & 12 & 14 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vec{y} \succeq \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ & \vec{y} \succeq \vec{0} \end{aligned}$$



The opt solution $x^* = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ satisfies

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

Any other feasible x satisfies

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

for some $w_1, w_2 \geq 0$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 14 \end{pmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 22 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= 22 - w_1 - w_2$$

y

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 12 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 22$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

In higher dimensions, take

$$\begin{aligned} \max \quad & c^T x & & c, x \in \mathbb{R}^n \\ \text{s.t.} \quad & Ax + w = b & & b, w \in \mathbb{R}^m \\ & x, w \geq 0 & & A \in \mathbb{R}^{m \times n} \end{aligned}$$

and suppose $\vec{x}, \vec{w}$ is an optimal point.

Further, suppose <u>exactly</u> n coord's of w are $=0$ and the rest are >0. (<u>with</u> loss of generality)