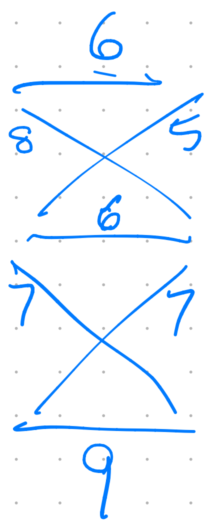
12 Sept 2025 — LP Duality



$$\min \quad 6x_{11} + 8x_{12} + 5x_{21} + 6x_{22} + 7x_{23} + 7x_{31} + 9x_{32}$$

$$\text{s.t.} \quad x_{11} + x_{12} = 1$$

$$x_{21} + x_{22} + x_{23} = 1$$

$$\vdots$$

$$x_{uv} \geq 0 \qquad \forall (u,v) \in E.$$

Let

$$x = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \end{bmatrix} \qquad \min \begin{bmatrix} 6 & 8 & 5 & 6 & 7 & 7 & 9 \end{bmatrix} x$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}}_{A} x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x \succeq 0$$

$$\overbrace{\begin{bmatrix} 8 & 7 & 8 & -2 & -1 & 0 \end{bmatrix}}^{y} A x = \begin{bmatrix} 8 & 7 & 8 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 20$$

If $Ax = \vec{1}$ then $\forall y \quad y^T A x = y^T \vec{1}$

This yields a lower bd on the optimum when $y^T A \leq [6\ 8\ 5\ 6\ 7\ 7\ 9]$.

To make the lower bound as large as possible, you'd solve

$$\text{maximize} \quad y^T \mathbb{1}$$
$$\text{subj to} \quad y^T A \leq [6\ 8\ 5\ 6\ 7\ 7\ 9]$$

Equiv.

$$\text{maximize} \quad \mathbb{1}^T y$$
$$\text{s.t.} \quad A^T y \leq \begin{bmatrix} 6 \\ 8 \\ 5 \\ 6 \\ 7 \\ 7 \\ 9 \end{bmatrix},$$

Weak LP Duality.
STRONG

For an optimization problem

$$\min \; c^T x$$

$$\text{s.t.} \; Ax \geq b \qquad \text{[primal]}$$

$$x \geq 0$$

If the minimum is well-defined and finite

$\{x \mid Ax \geq b, x \geq 0\} \neq \emptyset$ not $-\infty$

then it is ~~bounded below by~~ equal to

$$\text{max} \; \cancel{\sup} \; b^T y$$

$$\text{s.t.} \; A^T y \leq c \qquad \text{[dual]}$$

$$y \geq 0$$

Proof of weak duality.

Will show $\forall y$ satisfying $A^T y \preceq c$, $y \succeq 0$

and $\forall x$ satisfying $Ax \succeq b$, $x \succeq 0$

it holds that $b^T y \leq c^T x$.

WD follows by choosing y that maximizes $b^T y$, along with x that minimizes $c^T x$.

Assume $\boxed{A^T y \preceq c}$, $\boxed{y \succeq 0}$ $Ax \succeq b$, $x \succeq 0$

$y^T A \preceq c^T$ $\qquad$ $Ax - b \succeq 0$

Then

$$c^T x \geq y^T A x \geq y^T b = b^T y$$