$\boxed{\text{8 Sep 2025}}$ LP relaxation of bipartite matching

<u>Reminder.</u> An M-compatible labeling is a vector $y \in \mathbb{R}^V$:

① $c(u,v) \geq y_u + y_v \quad \forall$ edge (u,v)

② $c(u,v) = y_u + y_v \quad$ for $(u,v) \in M$

③ $y_u = \max_{w \in L} \{y_w\}$ for all $u \in L \cap F$

④ $y_v = \max_{w \in R} \{y_w\}$ for all $v \in R \cap F$.

<u>Algorithm (primal-dual)</u> for min-cost <u>perf matching</u>

Assume $c(u,v) \geq 0 \quad \forall$ edge (u,v).

$M = \emptyset$, $\vec{y} = \vec{0}$

while M is ~~not~~ a ~~perf~~ matching:

set $c^y(u,v) = c(u,v) - y_u - y_v \quad \forall$ edge (u,v)

// $= 0$ if $(u,v) \in M$

run Dijkstra's alg. to find shortest path from $L \cap F$ to $R \cap F$ w.r.t. costs $c^y(\cdot)$. Call it P.

update y using

distance from $L \cap F$ computed by Dijkstra.

$$y_u \leftarrow y_u + (c^y(P) - d_u)^+ \quad [u \in L]$$

$$y_v \leftarrow y_v - (c^y(P) - d_v)^{\boxed{+}} \quad [v \in R]$$

z^+ means $\max\{0, z\}$

update M to $M \oplus P$.

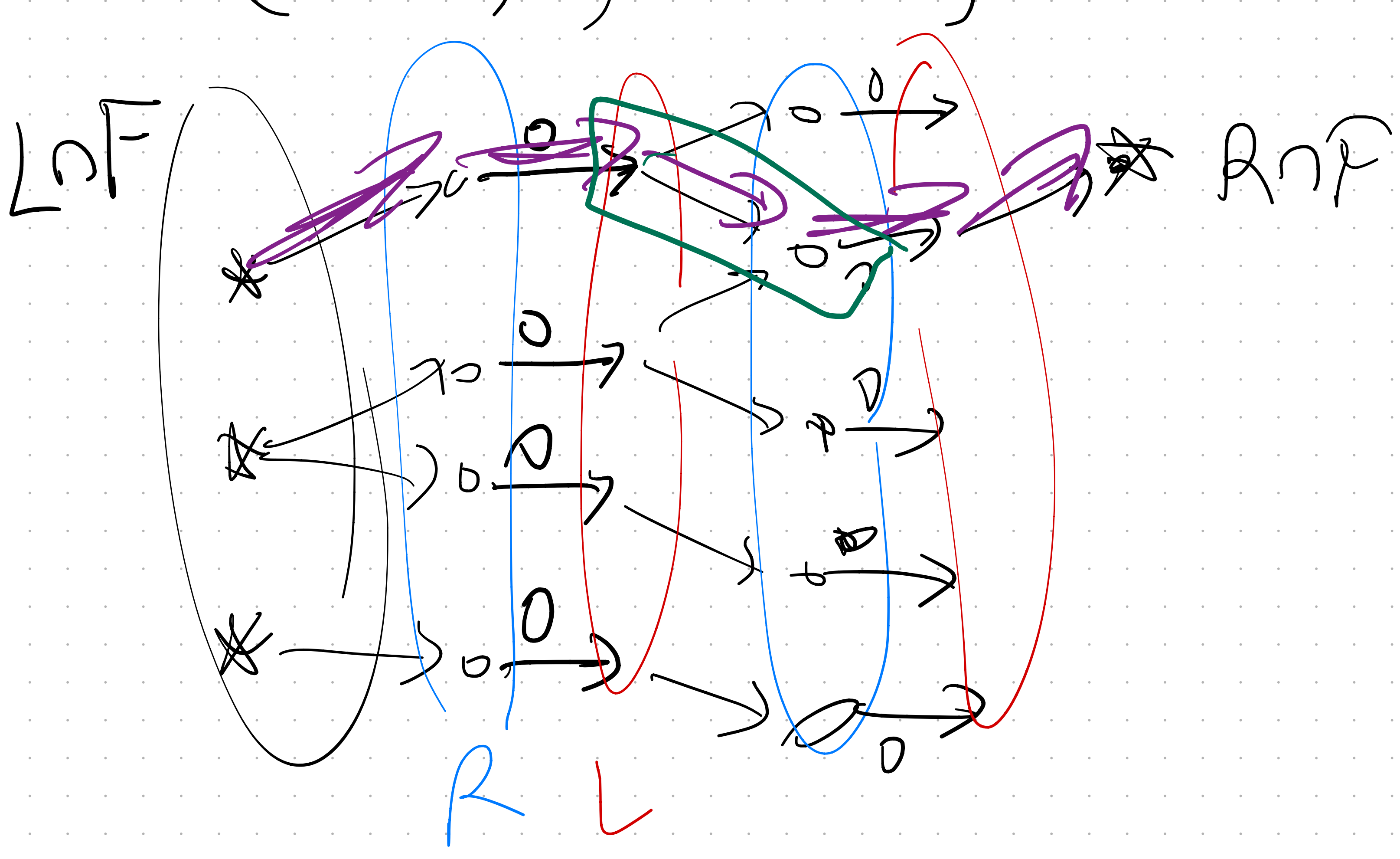
endwhile

output M.

We need to show the invariant

y is M-compatible

is preserved. Then correctness and $O(m\, n \log n)$ running time will follow.



For $(u,v) \in M$ $\quad c^y(v,u) = 0$

$$\Rightarrow \quad d_u = d_v$$

For $(u,v) \in P \setminus M$ $\quad c^y(u,v) \geq 0$

$$d_v = d_u + c^y(u,v)$$

$$= d_u + c(u,v) - y_u - y_v$$

$$(y_u - d_u) + (y_v + d_v) = c(u,v).$$

$$y_u^{new} = y_u + c^y(P) - d_u$$

$$y_v^{new} = y_v - \left(c^y(P) - d_v\right)$$

$$y_u^{new} + y_v^{new} = y_u + y_v$$

$$+ \cancel{c^y(P)} - \cancel{c^y(P)}$$

$$+ d_v - d_u$$

$$= (y_u - d_u) + (y_v + d_v)$$

$$= c(u,v).$$

For $e \notin (M \cup P)$ we need

$$y_u^{new} + y_v^{new} \leq c(u,v).$$

$\longrightarrow$ see typeset notes.

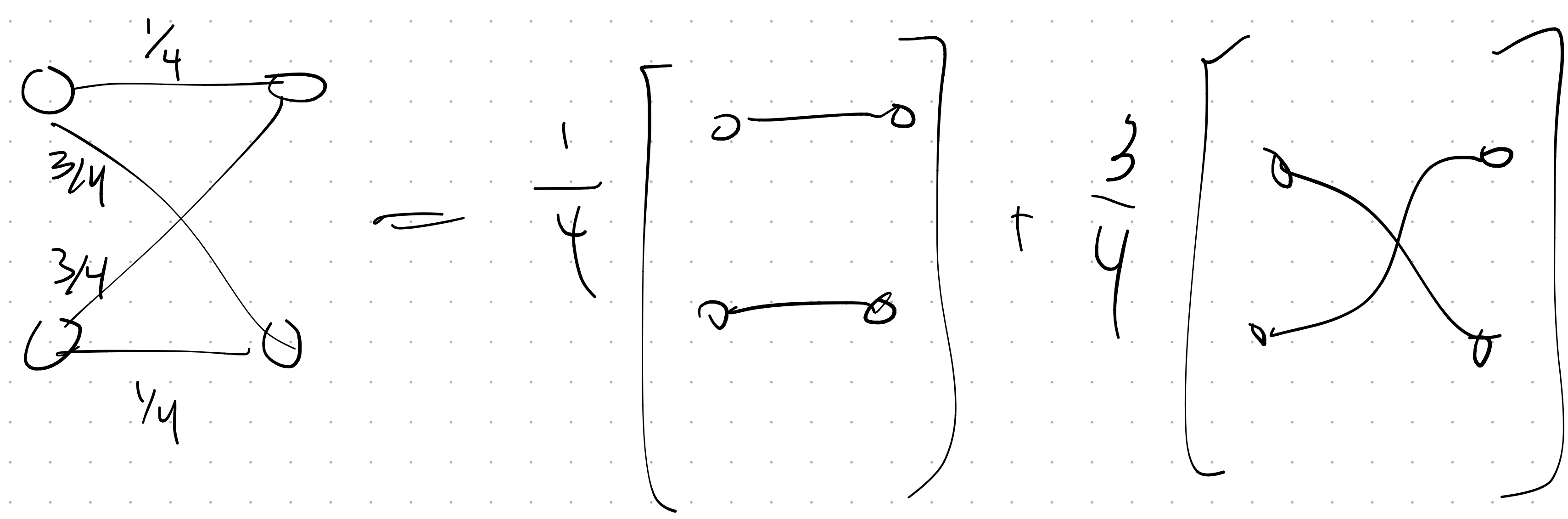
[§3.3 justifies the crazy formulas for y_u^{new}, y_v^{new}]

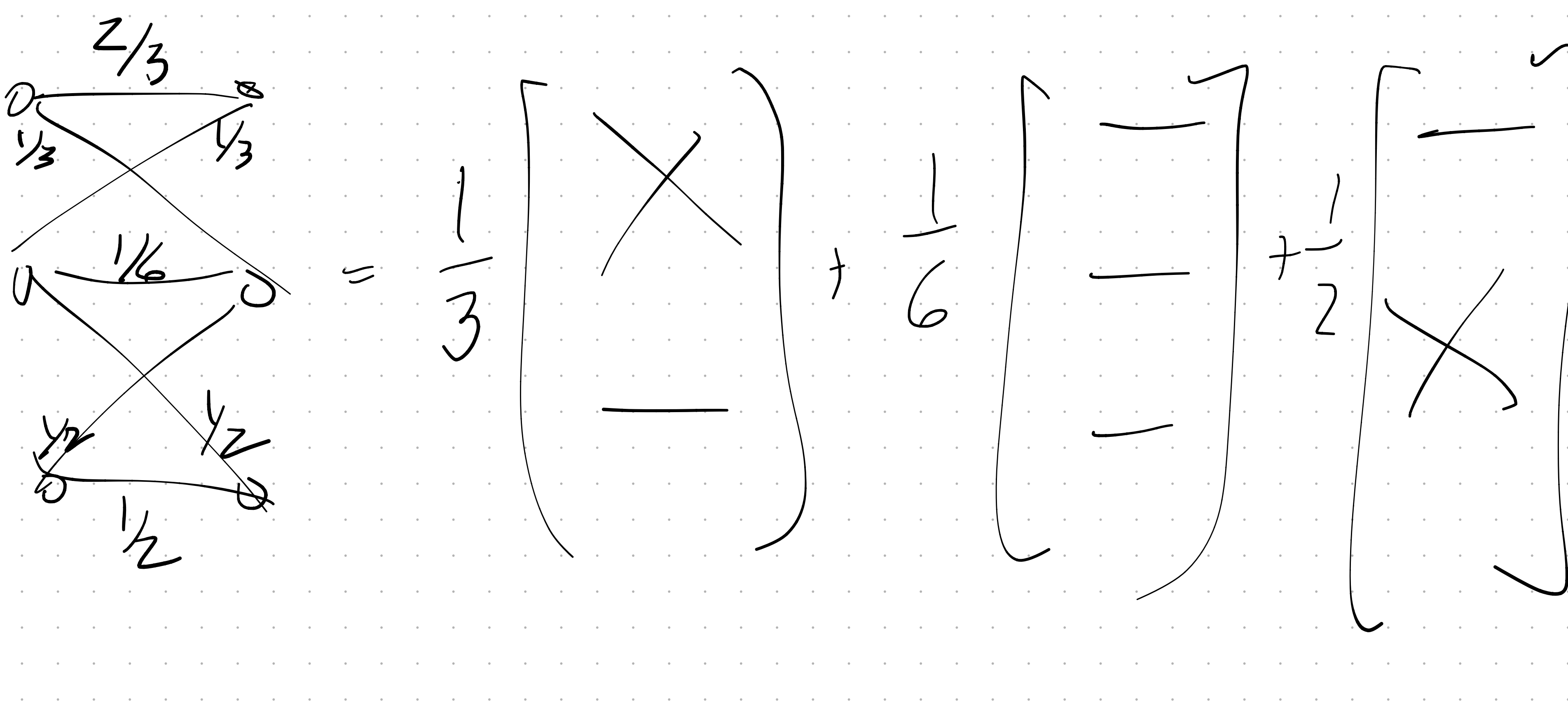
"Fractional perfect matching"

A labeling of edges (u,v) with numbers $x_{uv} \geq 0$ satisfying

$$\sum_{v \in R} x_{uv} = 1 \quad \forall u \in L$$

$$\sum_{u \in L} x_{uv} = 1 \quad \forall v \in R$$





Birkhoff - von Neumann Theorem

Every FPM in a bipartite graph is a convex combination of PMs.