$\boxed{\text{5 Sep 2025}}$ # Bipartite Min-Cost Perfect Matching

Bipartite $G = (V, E)$, $V = L \cup R$,

<u>assume</u> E has a perfect matching.
(At least one)

Edges have costs $c(u,v)$.

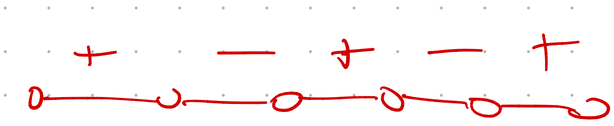
Cost of an edge set S,

$$c(S) = \sum_{(u,v) \in S} c(u,v)$$

Find perfect matching with min cost.

<u>Greedy "most improving path" algorithm:</u>

Initialize $M = \emptyset$.
while M is not a perfect matching
    find M-augmenting path P s.t.
        $c(M \oplus P)$ is minimized
    replace M with $M \oplus P$
endwhile
output M.



doesn't depend on P

alternating sum of edge costs on P

$$c(M \oplus P) = c(M) + \underbrace{c(P \setminus M) - c(P \cap M)}$$

minimize this using Bellman-Ford in each iteration.

RUNNING TIME $O(n \cdot mn) = O(mn^2)$.

Def. If G is a bipartite graph and M is a matching, an <u>M-compatible labeling</u> is an assignment of labels $y_v \in \mathbb{R}$ to vertices v s.t.

(i) $c(u,v) \geq y_u + y_v \quad \forall$ edge (u,v)

(ii) $c(u,v) = y_u + y_v$ if $(u,v) \in M$.

(iii) if $u \in L$ is free $y_u = \max\limits_{w \in L} \{y_w\}$.

(iv) if $v \in R$ is free $y_v = \max\limits_{w \in R} \{y_w\}$.

Fact 1. If an M-compatible labeling exists then M has the min cost among all matchings of its size.

Why? If M' is any other matching of same size as M

$$\text{cost}(M) = \sum_{(u,v) \in M} (y_u + y_v) = \sum_{u \in L} y_u + \sum_{v \in R} y_v - \sum_{w \text{ free in } M} y_w$$

$$\text{cost}(M') \geq \sum_{(u,v) \in M'} (y_u + y_v) = \sum_{u \in L} y_u + \sum_{v \in R} y_v - \sum_{w \text{ free in } M'} y_w$$

Reduced costs of edges.

Given matching M and compatible labeling y, let

$$c^y(u,v) = \begin{cases} c(u,v) - y_u - y_v & \text{if } (u,v) \notin M \\ 0 & \text{if } (u,v) \in M. \end{cases}$$

$$0 = y_u + y_v - c(u,v)$$

For path P in G_M how does $c^y(P)$ relate to $c(M \oplus P)$?

$$c^y(P) = \sum_{(u,v)\in P\setminus M} c(u,v) - y_u - y_v$$

$$c(M \oplus P) = c(M) + c(P \setminus M) - c(P \cap M)$$