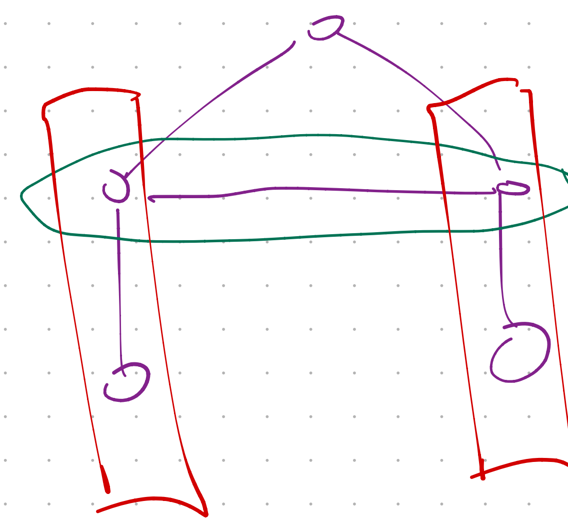
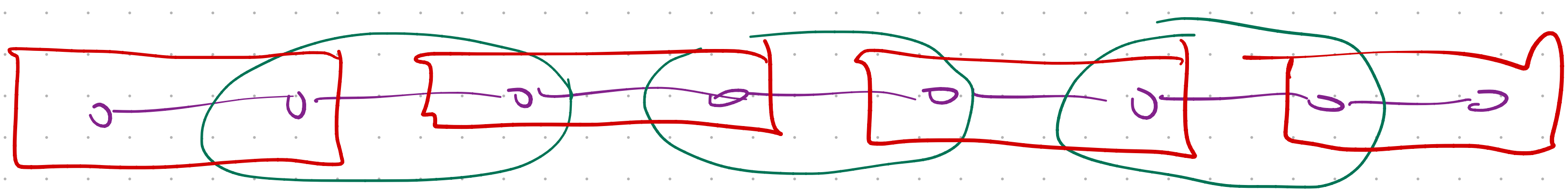
$\boxed{\text{27 Aug 2025}}$ Computing a Maximum Matching

Recap. Matching = set of edges with no vertices in common

Free = not part of any edge in the matching

Matched = not free





Def. An augmenting path in G with respect to M ("M-augmenting path") is a set of edges forming a (simple) path in G, starting & ending at free vertices, whose edges alternate btw belonging to M and not belonging.

Standard. "path" means "simple path" (no repeated vertices)

"walk": means "non-simple path" (may repeat vertices)

Lemma. For a graph G and a matching M, the following are equivalent.

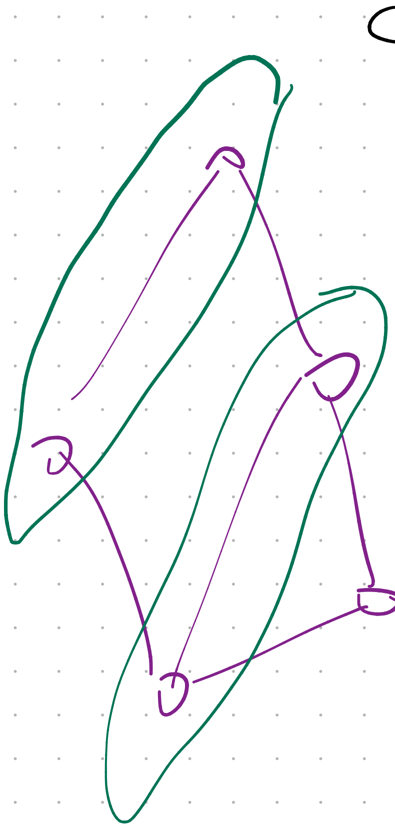
(A) M is a maximum matching in G

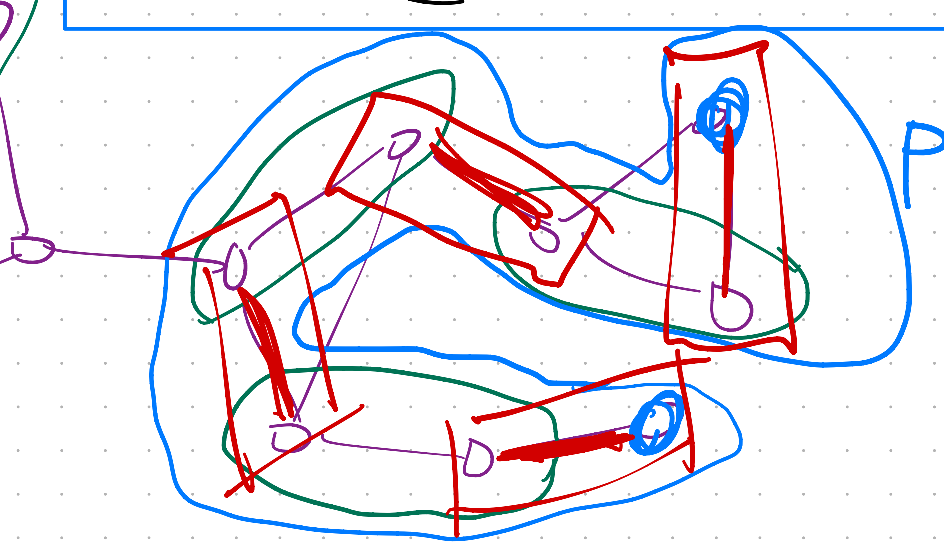
(B) G contains no M-augmenting path.

Proof. $(\neg B) \Longrightarrow (\neg A)$

If P is a M-augmenting path construct new edge set M' by

$$\boxed{M' = \left[M \setminus (P \cap M)\right] \cup (P \setminus M)}$$





P

$|M'| = |M| + 1.$

$M' = M \oplus P$

$(\neg A) \Rightarrow (\neg B)$

"If $\exists$ a matching M' with more edges than M, then $\exists$ a M-augmenting path."

Look at $M \oplus M'$. It has maximum degree 2, so it is a union of disjoint paths and cycles.

The edges of each of these paths/cycles alternate between M and M'.

M' has more edges than M.

$\Longrightarrow$ At least one of these paths/cycles has more M' edges than M edges.

$\Rightarrow$ It must be a path P that starts and ends with M' edges.

This P is an M-augmenting path.

Endpoints of P belong to an edge of $M' \cap P$. They can't belong to any <u>other</u> edge of M'. They also don't belong to any other edge of $M \oplus M'$. This implies they can't belong to any edge of $M \Longrightarrow$ they are free w.r.t. M.

Naive Iterative Max Matching

<u>NIMM(G):</u>

Start with $M = \emptyset$.
while $\exists$ an M-aug path P
  $M \leftarrow M \oplus P$
endwhile
output M.

Lemma above $\Rightarrow$ if this terminates its output is a max matching.

while-loop iterations is $\leq$ # edges in G

so termination is assured.

How fast can we test if $\exists$ an M-augmenting path?