22 Nov 2021 Spectral Sparsification

Reduce # edges of a graph from $O(n^2)$ down to $O(n \log n)$, approximately preserving its Laplacian.

Def. A, B symmetric pos semidef matrices, the relation

$$A \preccurlyeq B$$

means $\forall x \in \mathbb{R}^n \quad \langle x, Ax \rangle \leq \langle x, Bx \rangle$.
Since that is equiv't to $\langle x, (B-A)x \rangle \geq 0$ the relation $A \preccurlyeq B$ is equiv to:

$B-A$ is pos. semidefinite matrix.

Ex. If A, B are diagonal matrices, then $A \preccurlyeq B$ holds iff $a_{ii} \leq b_{ii} \; \forall i \in [n]$.

Def. We say A is a (multiplicative) ε-approximation to B if

$$(1-\varepsilon) B \preccurlyeq A \preccurlyeq (1+\varepsilon) B$$

<u>Obs.</u> If G, H are weighted graphs and L_H is a multiplicative ε-approx to L_G then $\forall\ \emptyset \subsetneq S \subsetneq V$

$$(1-\varepsilon)\, w_G(\partial S) \leq w_H(\partial S) \leq (1+\varepsilon)\, w_G(\partial S)$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ (1-\varepsilon)\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle & & \langle \mathbb{1}_S, L_H \mathbb{1}_S \rangle & & (1+\varepsilon)\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle \end{array}$$

<u>Idea:</u> For any vertex v let $e_v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow v$.

For vertices $u \neq v$ let $\delta_{uv} = e_u - e_v$.

Then

$$\delta_{uv}\delta_{uv}^T = \begin{array}{cc} \begin{bmatrix} 0 & & 0 & & 0 \\ & 1 & & -1 & \\ 0 & & 0 & & 0 \\ & -1 & & 1 & \\ 0 & & 0 & & 0 \end{bmatrix} & \begin{matrix} \\ u \\ \\ v \\ \\ \end{matrix} \\ \begin{matrix} \quad u \qquad v \quad \end{matrix} & \end{array}$$

= Laplacian of a weight-1 edge from u to v.

$$L_G = \sum_{(u,v) \in E} w(u,v)\, \delta_{uv}\delta_{uv}^T.$$

To be designed..

Initialize $w_H(u,v)=0$ $\forall u,v$.

Idea: Let π be a prob distrib on edges. Draw k samples from π, $\{(u_j,v_j)\}_{j=1}^{k}$, and each time (u,v) is sampled add $\hat{w}(u,v)$ to its weight in H.

with replacement

$$\mathbb{E}[L_H] = k \cdot \sum_{(u,v)\in E} \hat{w}(u,v)\cdot \pi(u,v)\cdot \delta_{uv}\delta_{uv}^T$$

Design π and $\hat{w}(\cdot,\cdot)$ with the following objectives in mind.

① L_H should be an unbiased estimate of L_G:

$$\mathbb{E}[L_H] = L_G$$

$\forall u,v$ $\quad k\cdot \hat{w}(u,v)\cdot \pi(u,v) = w(u,v)$

$$\hat{w}(u,v) = \frac{w(u,v)}{k\cdot \pi(u,v)}$$

② For $k = O(n \log n)$, it should hold (with const. probability) that

$$(1-\varepsilon)L_G \preccurlyeq L_H \preccurlyeq (1+\varepsilon)L_G$$

To accomplish Step 2 we need a version of the Chernoff bound for independent matrix-valued random vars.

<u>Theorem</u> (Ahlswede–Winter Ineq.) Suppose $X_1, \ldots, X_k$ are indep. random vars taking values in $n \times n$ symmetric positive semidefinite matrices. Let

$$\overline{X} = \frac{1}{k}(X_1 + \ldots + X_k)$$

$$U = \mathbb{E}\,\overline{X}.$$

If R is a scalar such that $\Pr(X_i \preccurlyeq RU) = 1 \;\; \forall i$ then

$$\Pr(\overline{X} \text{ is a mult. } \varepsilon\text{-approx to } U)$$

$$\geq 1 - 2n \cdot \exp\left(-\frac{\varepsilon^2 k}{4R}\right)$$

For our random matrix L_H, we can set

$$X_j = \hat{w}(u_j, v_j)\, \delta_{u_j v_j} \delta_{u_j v_j}^T.$$

Then
$$L_H = \sum_{j=1}^{k} X_j.$$
$$\overline{X} = \frac{1}{k} L_H$$
$$\mathcal{U} = \frac{1}{k} L_G$$

We aim to use A.-W. ineq. to reason about
$$\Pr\left((1-\varepsilon)\mathcal{U} \preceq \overline{X} \preceq (1+\varepsilon)\mathcal{U} \right).$$

We just need a constant R such that $X_j \preceq R \cdot \mathcal{U} = \frac{R}{k} L_G$ for all j.

$$X_j = \hat{w}(u,v) \cdot \delta_{uv}\delta_{uv}^T = \frac{1}{k} \frac{w(u,v)}{\pi(u,v)} \delta_{uv}\delta_{uv}^T.$$

For $X_j \preceq \frac{R}{k} L_G$ to hold, we need
$$\cancel{\frac{1}{k}} \frac{w(u,v)}{\pi(u,v)} \delta_{uv}\delta_{uv}^T \preceq \cancel{\frac{1}{k}} R L_G$$
$$w(u,v)\, \delta_{uv}\delta_{uv}^T \preceq R\, \pi(u,v)\, L_G$$

Observe $w(u,v)\, \delta_{uv}\delta_{uv}^T \preceq L_G$ always holds.

So setting $\pi(u,v) = \frac{1}{|E|}$ and $R = |E| = m$ satisfies the criteria for Ahls-Wint.

$$\Rightarrow \Pr(\varepsilon\text{-approx}) \geq 1 - 2n \exp\left(-\frac{\varepsilon^2 k}{4m}\right)$$

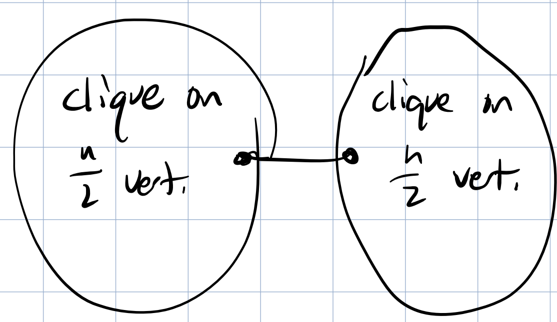
Want LHS $\geq \frac{1}{2}$, so we want

$$\exp\left(\frac{\varepsilon^2 k}{4m}\right) \geq 4n$$

$$\frac{\varepsilon^2 k}{4m} \geq \ln(4n)$$

$$k \geq 4m\,\varepsilon^{-2} \ln(4n).$$

So π = uniform distrib on edges is a bad idea: k too large to be useful.



All edges weight 1.

§7 of spectral graph alg notes: gives the right sampling procedure: probability proportional to <u>effective</u> <u>resistance</u>.