$\boxed{\text{19 Nov 2021}}$ Cheeger's Inequality

For edge-weighted graph G, and vertex set S, $\emptyset \subsetneq S \subsetneq V$,

$$\phi(S) = \frac{w(\partial S) \cdot d(V)}{d(S) \cdot d(V \setminus S)}$$

$$h(S) = \frac{w(\partial S)}{\min\{d(S), d(V \setminus S)\}}$$

$$\frac{1}{2}\phi(S) \leq h(S) \leq \phi(S).$$

Problem: Given G, find S that minimizes $\phi(S)$.

Edge-weighted clique, H,

$$w_H(u,v) = \frac{d(u) \cdot d(v)}{d(V)}.$$

We saw $\forall S$, $\emptyset \subsetneq S \subsetneq V$,

$$\phi(S) = \frac{\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle}{\langle \mathbb{1}_S, L_H \mathbb{1}_S \rangle} \geq \frac{\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle_D}{\langle \mathbb{1}_S, \mathbb{1}_S \rangle_D}$$

where $\langle x, y \rangle_D = \sum_v d(v) x_v y_v$

$$\bar{L}_G = D_G^{-1} L_G$$

D_G = diagonal matrix with vertex degrees on diagonal.

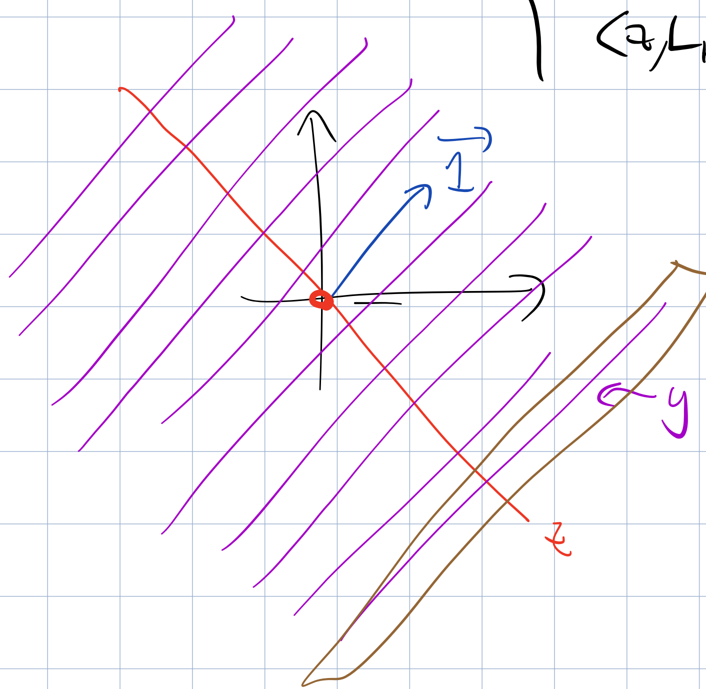
Lemma. $\lambda_2(\bar{L}_G)$ is the minimum of $\dfrac{\langle y, L_G y\rangle}{\langle y, L_H y\rangle}$ over all $y \in \mathbb{R}^V$ not parallel to $\vec{1}$.

Proof.

Courant–Fisher (b/c $D_G^{-1} L_G$ is self-adj. w.r.t. $\langle , \rangle_D$)

$$\lambda_2(\bar{L}_G) = \min \left\{ \frac{\langle z, D_G^{-1} L_G z\rangle_D}{\langle z, z\rangle_D} \;\middle|\; \begin{array}{l}\langle z, \vec{1}\rangle_D = 0 \\ z \neq 0\end{array} \right\}$$

$$= \min \left\{ \frac{\langle z, L_G z\rangle}{\langle z, L_H z\rangle} \;\middle|\; \begin{array}{l}\langle z, \vec{1}\rangle_D = 0 \\ z \neq 0\end{array} \right\}$$



ratio $\dfrac{\langle y, L_G y\rangle}{\langle y, L_H y\rangle}$ is constant along each of these lines.

Wherever the minimum over y is attained, there must be a $z = y + s\cdot\vec{1}$ such that $\langle z, \vec{\mathbb{1}} \rangle_D = 0$ and the minimum is also attained at z.

<u>Corollary</u>: $\phi(S) \geq \lambda_2(\bar{L}_G)$ for all $\emptyset \subsetneq S \subsetneq V$

<u>Proof</u>:

$$\phi(S) = \frac{\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle}{\langle \mathbb{1}_S, L_H \mathbb{1}_S \rangle} \geq \min_y \left\{ \frac{\langle y, L_G y \rangle}{\langle y, L_H y \rangle} \right\}$$

$$= \lambda_2(\bar{L}_G).$$

<u>Theorem</u> (Cheeger's Inequality)

Let $\phi(G) = \min\{ \phi(S) \mid \emptyset \subsetneq S \subsetneq V \}$

Then

$$\lambda_2(\bar{L}_G) \leq \phi(G) \leq \sqrt{8 \lambda_2(\bar{L}_G)}.$$

<u>Proof</u>: First inequality is the corollary above.

Second inequality: we'll show, for any $y \in \mathbb{R}^V$ not parallel to $\vec{\mathbb{1}}$, how to sample a random vertex set S, such that if

$$Q(y) = \frac{\langle y, L_G y \rangle}{\langle y, L_H y \rangle}$$

then

$$\mathbb{E}[w(\partial S)] \leq \sqrt{2 Q(y)} \cdot \langle y, y \rangle_D$$

$$\mathbb{E}\left[\min\{d(S), d(V\setminus S)\}\right] \geq \langle y, y \rangle_D$$

From those two ineq's it follows that the support of the distrib of S contains a set S_0 with

$$h(S_0) \leq \sqrt{2\,Q(y)}$$

$$\phi(S_0) \leq 2\,h(S_0) \leq \sqrt{8Q(y)}\,.$$

The theorem will follow by taking y to be any vector that minimizes $Q(y)$ over $y \not\parallel \vec{1}$.

To sample S we will be selecting a random threshold Θ and defining $S = \{v \mid y_v < \theta\}$.

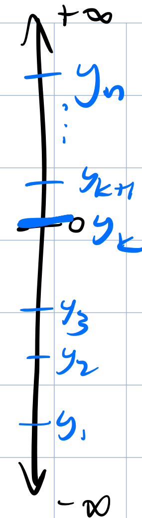
Note $Q(y) = \dfrac{\langle y, L_G y\rangle}{\langle y, L_H y\rangle}$ is invariant under scaling and $y \mapsto y + s\cdot\vec{1}$.

So w.l.o.g. we can assume

(i) The "median coordinate" of y is $\emptyset$.

$$\sum_{u: y_u<0} d(u) \le \sum_{u: y_u \ge 0} d(u)$$

$$\sum_{u: y_u \le 0} d(u) \ge \sum_{u: y_u>0} d(u)$$



(2.) $y_1^2 + y_n^2 = 1$.

Let

$$z_i = \begin{cases} -y_i^2 & \text{if } y_i < 0 \\ y_i^2 & \text{if } y_i > 0 \end{cases}$$

By assumption 2, $z_n - z_1 = 1$.

Sample t uniformly random in $[z_1, z_n]$, set

$$S = \{v \mid z_v < t\}.$$

<u>Step 1.</u> Bound $\mathbb{E}[\min(d(S), d(V \setminus S))]$.

$d(v)$ appears in this expression if and only if threshold t

is between 0 and z_v.

Hence $\Pr(t \text{ between } 0 \text{ and } z_v)$

$$= |z_v| = y_v^2.$$

$$\mathbb{E}[\min(d(S), d(V\setminus S))] = \sum_v d(v)\, y_v^2$$

$$= \langle y, y \rangle_D.$$

Step 2. $\mathbb{E}[w(\partial S)] = \sum_{(u,v)\in E} w(u,v) \cdot \Pr((u,v) \in \partial S)$

$$= \sum_{(u,v)\in E} w(u,v) \Pr(t \text{ between } z_u, z_v)$$

$$= \sum_{(u,v)\in E} w(u,v)\, |z_u - z_v|$$

$\vdots$ (something using Cauchy-Schwartz)

$$\leq \sqrt{2\, Q(y) \, \langle y, y \rangle_D}$$