$\boxed{\text{17 Nov 2021}}$ The Laplacian of a Graph

All graphs will be non-negative edge weighted.

"Graph" is really just a finite vertex set V and a symmetric edge-weight function

$$w : V \times V \longrightarrow [0, \infty)$$

satisfying $w(u,v) = w(v,u)$, $w(v,v) = 0$ $\forall u,v$.

The Laplacian matrix of G, L_G, is defined as follows:

$$(L_G)_{uv} = \begin{cases} -w(u,v) & \text{if } u \neq v \\ d(v) & \text{if } u = v \end{cases}$$

where $d(v) \triangleq \sum_u w(u,v)$.

The <u>Laplacian quadratic form</u> of G is the quadratic function $\mathbb{R}^V \to \mathbb{R}$ defined by $x \mapsto \langle x, L_G x \rangle$.

$$\langle x, L_G x \rangle = \sum_{v \in V} x_v \cdot (L_G x)_v$$

$$= \sum_{v \in V} x_v \cdot \left(\sum_{u \in V} (L_G)_{vu} x_u \right)$$

$$= \sum_{(u,v) \in V \times V} (L_G)_{vu} \; x_u x_v$$

$$= \sum_{v} \left(\sum_{u \in V} w(u,v) \right) x_v^2$$

$$- \sum_{v} \sum_{u \neq v} w(u,v) \, x_u x_v$$

$$= \frac{1}{2} \sum_{v} \sum_{u \neq v} w(u,v) \left[x_u^2 + x_v^2 - 2 x_u x_v \right]$$

$$= \frac{1}{2} \sum_{v} \sum_{u \neq v} w(u,v) \left(x_u - x_v \right)^2 .$$

<u>Observation:</u> $\langle x, L_G x \rangle \geq 0 \quad \forall x$.

It can only equal zero when $x_u = x_v$ for all (u,v) with $w(u,v) > 0$.

In other words $\lambda_1(L_G) = 0$, and the 0-eigenspace consists of vectors x that are constant on each connected component of G.

Partitioning G using a sparse cut.

Def. For a vertex set S let

$$d(S) = \sum_{v \in S} d(v)$$

$$\partial S = \left\{ (u,v) \;\middle|\; u \in S,\ v \notin S \right\}$$

$$w(\partial S) = \sum_{(u,v) \in \partial S} w(u,v).$$

$$\phi(S) = \frac{w(\partial S) \cdot d(V)}{d(S) \cdot d(V \setminus S)}.$$

$$h(S) = \frac{w(\partial S)}{\min\{d(S), d(V \setminus S)\}}$$

$$= \frac{w(\partial S) \cdot \max\{d(S), d(V \setminus S)\}}{d(S) \cdot d(V \setminus S)}$$

$$= \phi(S) \cdot \max\{d(S), d(V \setminus S)\} / d(V)$$

$$= \phi(S) \cdot \frac{\max\{d(S), d(V \setminus S)\}}{d(S) + d(V \setminus S)}$$

$$\in \left[\tfrac{1}{2} \phi(S), \; \phi(S) \right].$$

Observation: If $\mathbb{1}_S$ denotes the vector

$$(\mathbb{1}_S)_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases}$$

then

$$\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle$$

$$= \frac{1}{2} \sum_u \sum_{v \neq u} w(u,v) \left((\mathbb{1}_S)_u - (\mathbb{1}_S)_v \right)^2$$

$$= w(\partial S)$$

The other terms in ϕ also have a nice Laplacian interpretation.

Define H to be the weighted graph with vertex set V and weights

$$w_H(u,v) = \frac{d(u) \cdot d(v)}{d(V)}.$$

Then

$$\langle \mathbb{1}_S, L_H \mathbb{1}_S \rangle = \frac{1}{2} \sum_u \sum_{v \neq u} w_H(u,v) \left((\mathbb{1}_S)_u - (\mathbb{1}_S)_v \right)^2$$

$$= w_H(\partial S)$$

$$= \frac{1}{d(V)} \sum_{u \in S} \sum_{v \notin S} d(u) \cdot d(v)$$

$$= \frac{1}{d(V)} \left(\sum_{u \in S} d(u) \right) \left(\sum_{v \notin S} d(v) \right)$$

$$= \frac{d(S) \cdot d(V \setminus S)}{d(V)}.$$

Hence...

$$\phi(S) = \frac{w(\partial S) \cdot d(V)}{d(S) \cdot d(V \setminus S)} = \frac{\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle}{\langle \mathbb{1}_S, L_H \mathbb{1}_S \rangle}.$$

This almost looks like a Rayleigh quotient.

Define a different inner product:

$$\langle x, y \rangle_D = \sum_{v \in V} d(v) \, x_v y_v.$$

If G has no isolated vertices, i.e. $d(v) > 0 \quad \forall v$, then

$\langle x, y \rangle_D$ is a positive ~~definite~~ inner product.

<u>Lemma</u>: For all $x \in \mathbb{R}^V$,

$$\langle x, L_H x \rangle \leq \langle x, x \rangle_D$$

with equality if and only if $\langle x, \mathbb{1} \rangle_D = 0$.

<u>Proof</u>. Let $p(v) = d(v) / d(V)$.

The $p(v)$ are positive numbers summing to 1, i.e. $p(\cdot)$ is a prob distrib on V.

$$\frac{1}{d(V)} \langle x, L_H x \rangle = \frac{1}{2 \cdot d(V)^2} \sum_u \sum_{v \neq u} d(u) d(v) (x_u - x_v)^2$$

$$= \frac{1}{2} \sum_u \sum_{v \neq u} p(u) p(v) (x_u - x_v)^2$$

$$= \frac{1}{2} \sum_u \sum_v p(u) p(v) (x_u - x_v)^2$$

$$= \frac{1}{2} \mathbb{E}_{u,v \sim p} \left[(x_u - x_v)^2 \right]$$

$$= \frac{1}{2} \mathbb{E}_{u \sim p}\left[x_u^2\right] + \frac{1}{2} \mathbb{E}_{v \sim p}\left[x_v^2\right] - \mathbb{E}_{u,v \sim p}\left[x_u x_v\right]$$

$$= \mathbb{E}_{u \sim p}\left[x_u^2\right] - \left(\mathbb{E}_{u \sim p}\left[x_u\right]\right)^2$$

$$\frac{1}{d(V)} \langle x, x \rangle_D = \frac{1}{d(V)} \sum_{u \in V} d(u)\, x_u^2$$

$$= \sum_{u \in V} p(u)\, x_u^2$$

$$= \mathbb{E}_{u \sim p}\left[x_u^2\right].$$

So

$$\frac{1}{d(V)} \langle x, L_H x \rangle \leq \frac{1}{d(V)} \langle x, x \rangle_D$$

Equality iff

$$0 = \mathbb{E}_{u \sim p}\left[x_u\right]$$

$$= \frac{1}{d(V)} \sum_{u \in V} d(u)\, x_u$$

$$= \frac{1}{d(V)} \langle x, \mathbb{1} \rangle_D .$$

Recall: $\phi(S) = \dfrac{\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle}{\langle \mathbb{1}_S, L_H \mathbb{1}_S \rangle} \geq \dfrac{\langle \mathbb{1}_S, L_G \mathbb{1}_S \rangle}{\langle \mathbb{1}_S, \mathbb{1}_S \rangle_D}$

$$= \frac{\langle \mathbb{1}_S, D_G^{-1} L_G \mathbb{1}_S \rangle_D}{\langle \mathbb{1}_S, \mathbb{1}_S \rangle}$$

where D_G = diagonal matrix with $d(v)$ in the (v,v) diag entry.

Observe $\langle x, y \rangle_D = \sum d(v) x_v y_v$

$$= \langle x, D_G y \rangle.$$

Def. $D_G^{-1} L_G$ is the <u>normalized</u> <u>Laplacian</u> of G. It is self-adjoint w.r.t. the $\langle \, , \rangle_D$ inner prod.

$$\langle x, D_G^{-1} L_G y \rangle_D = \langle x, L_G y \rangle$$

$$= \langle L_G x, y \rangle$$

$$= \langle D_G^{-1} L_G x, y \rangle_D$$

verifies L_G is self adjoint