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Spectral Graph Theory Intro: The Courant Fischer Theorem

The Courant-Fischer Theorem

Setup: V a finite (n) dimensional inner product space over $\mathbb{R}$.

i.e. $\langle x,y\rangle$ operation taking values in $\mathbb{R}$,

- linear in x and y
- symmetric $\langle x,y\rangle = \langle y,x\rangle$
- ~~pos~~ definite

$\langle x,x\rangle \geq 0$ with equality only when $x=0$.

e.g. $\mathbb{R}^n$ with $\langle x,y\rangle = x\cdot y$.

$A: V \to V$ a self-adjoint linear transformation.

$$\langle Ax,y\rangle = \langle x, Ay\rangle \quad \forall x,y.$$

i.e. if $\langle\, ,\rangle$ is dot product, the matrix representing A is symmetric.

Def. Linear subspace $W \subset V$ is <u>A-invariant</u> if $Aw \in W$ for all $w \in W$.

Lem 1. If A self-adjoint and W is A-invt, then $W^\perp \triangleq \{ v \mid \forall w \in W \ \langle v, w \rangle = 0 \}$ is also A-invariant.

Proof. If $v \in W^\perp$ we must show $Av \in W^\perp$, i.e. for all $w \in W$ we must show

$$\langle Av, w \rangle = 0$$

Self-adjointness of A implies

$$\langle Av, w \rangle = \langle v, Aw \rangle \quad \leftarrow \text{belongs to } W \text{ b/c } W \text{ is } A\text{-invt.}$$

$$= 0 \quad \leftarrow \text{because } v \in W^\perp.$$

Def. The Rayleigh quotient of $x \in V$ with respect to $A : V \to V$ is defined, for $x \neq 0$, as

$$RQ_A(x) = \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

Lem 2. If $A : V \to V$ self-adjoint then

$$\lambda_1(A) = \inf \{ RQ_A(x) \mid x \neq 0 \}$$

is achieved at some $x \in V$ and $\lambda_1(A)$ is an eigenvalue of A and x is an eigenvector.

("Variational characterization of eigenvalues")

<u>Proof.</u> $RQ_A(x) = RQ_A(tx) \quad \forall t \neq 0$

so

$$\inf\{RQ_A(x) \mid x \neq 0\} = \inf\{RQ_A(x) \mid \langle x,x\rangle = 1\}.$$

$\{x \mid \langle x,x\rangle = 1\}$ is compact so the infimum is achieved at some x.

Since RQ_A is minimized at x, ∇RQ_A vanishes at x.

$$RQ_A(x+\varepsilon y) = RQ_A(x) + \langle \nabla RQ_A, y\rangle \varepsilon + O(\varepsilon^2).$$

$$\| $$

$$\frac{\langle x+\varepsilon y,\ Ax + A\varepsilon y\rangle}{\langle x+\varepsilon y, x+\varepsilon y\rangle} \qquad \langle x, Ay\rangle = \langle Ax, y\rangle$$

$$\| $$

$$\frac{\langle x, Ax\rangle + \varepsilon \underbrace{\langle y, Ax\rangle}_{=\langle Ax, y\rangle} + \varepsilon \langle x, Ay\rangle + O(\varepsilon^2)}{\langle x,x\rangle + 2\varepsilon\langle x,y\rangle + O(\varepsilon^2)}.$$

$$= \frac{\langle x, Ax\rangle + 2\varepsilon\langle Ax, y\rangle + O(\varepsilon^2)}{\langle x, x\rangle + 2\varepsilon\langle x, y\rangle + O(\varepsilon^2)}$$

When $\langle x, y\rangle = 0$ the coefficient of ε is $2\langle Ax, y\rangle$.
This cannot be non-zero.
(Otherwise one of $RQ_A(x+\varepsilon y)$ or $RQ_A(x-\varepsilon y)$ would be less than $RQ_A(x)$ for small ε.)

$\Longrightarrow$ whenever $\langle x, y\rangle = 0$, then $\langle Ax, y\rangle = 0$

$\Longrightarrow$ $Ax = \lambda x$ for some λ.

So x is an eigenvector of A.

$$RQ_A(x) = \frac{\langle x, Ax\rangle}{\langle x, x\rangle} = \lambda\frac{\langle x, x\rangle}{\langle x, x\rangle} = \lambda$$

$\Rightarrow$ $RQ_A(x)$ is an eigenvalue of A.

<u>Corollary</u> $\lambda_n(A) \triangleq \sup\{RQ_A(x) \mid x \neq 0\}$ is also an eigenvalue, the supremum is attained at an eigenvector.

<u>Proof.</u> Use lemma with matrix $-A$, observing that $RQ_{-A}(x) = -RQ_A(x)$.

Theorem. (Courant - Fischer) Let $A : V \to V$ be self-adjoint, V finite dimensional, $n = \dim V$.

(1) A has n real eigenvalues (counting multiplicities). Denote them

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A).$$

(2) $$\lambda_k(A) = \min_{\substack{\dim W = k \\ W \subset V}} \max \left\{ RQ_A(x) \,\middle|\, x \in W, x \neq 0 \right\}.$$

(3) $$\lambda_{n+1-k}(A) = \max_{\substack{\dim W = k \\ W \subset V}} \min \left\{ RQ_A(x) \,\middle|\, x \in W, x \neq 0 \right\}$$

Proof. Induction on i. Let $W_0 = \{0\}$; $V_0 = V$. Induction hypothesis is that W_i, V_i are A-invariant, $V_i = W_i^\perp$, W_i is i-dimensional.

If $i < n$, V_i is $(n-i)$-dimensional and A-invariant, so Lemma 2 says $\inf \left\{ RQ_A(x) \,\middle|\, x \in V_i, x \neq 0 \right\}$ is attained at an eigenvector of A.

Let $\lambda_{i+1}(A)$ denote this eigenvalue,

x_{i+1} denote the eigenvector,

W_{i+1} = span of W_i and x_{i+1}.

$V_{i+1} = W_{i+1}^{\perp}$.

W_{i+1} and V_{i+1} are A-invariant,

$\dim W_{i+1} = i+1$.

This inductive construction produces n lin indep eigenvectors

$x_1, x_2, \ldots, x_n$.

$\therefore$ A has n eigenvalues counting multiplicity.

$\lambda_i(A) \leq \lambda_{i+1}(A)$ b/c $\lambda_i(A) = \min\{RQ_A(x) \mid x \in V_i\}$

$\lambda_{i+1}(A) = \min\{RQ_A(x) \mid x \in V_{i+1}\}$

$V_{i+1} \subset V_i$.

Why is $\lambda_k(A) = \min\limits_{\dim W = k} \max\limits_{\substack{x \neq 0 \\ x \in W}} RQ_A(x)$?

First note $\dim W_k = k$ and $A: W_k \to W_k$ has eigenvalues $\lambda_1(A) \leq \ldots \leq \lambda_k(A)$.

so $\max\{RQ_A(x) \mid x \in W_k, x \neq 0\} = \lambda_k(A)$.

If W' is some other k-dimensional subspace, then $W' \cap V_{k-1}$ is positive-dimensional.

So there is a vector $x \neq 0$, $x \in W' \cap V_{k-1}$.

$$V_{k-1} = \operatorname{span}(x_k, x_{k+1}, \ldots, x_n)$$

min eigenvalue of $A : V_{k-1} \to V_{k-1}$ is $\lambda_k(A)$.

$$\Longrightarrow \quad RQ_A(x) \geq \lambda_k(A).$$

$$\Longrightarrow \max_{\substack{x \in W' \\ x \neq 0}} \{ RQ_A(x) \} \geq \lambda_k(A).$$

Proof of Courant-Fischer for max min $RQ_A(x)$ is symmetric.