10 Nov 2021

Multicommodity flow & Sparsest cut

Multicommodity flow (MCF): Given graph $G=(V,E)$ and set of k sourse-sink pairs $\{(s_i,t_i)\}_{i=1}^{k}$. Edges have capacities $c(u,v)$.

Feasible MCF: A flow value $f_P \geq 0$ for each $P \in Paths(s_i,t_i)$ for each $i \in [k]$, satisfying the capacity constraints

$$\sum_{P:\, e \in P} f_P \leq c(e) \quad \forall e \in E.$$

edges could be directed or undirected

for dir graphs, sum over paths that use $e=(u,v)$ going from u to v

for undir, sum over paths that use e in either direction

What do you want to maximize?

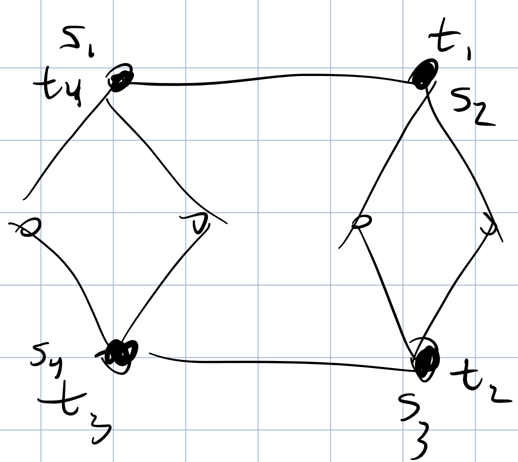
Throughput: $\sum_p f_p$.

Concurrent flow rate: $\min_{i \in [k]} \left\{ \sum_{p \in Paths(s_i, t_i)} f_p \right\}$.

(corresponds to fairly sharing the network among k commodities)

Max-flow min-cut theorem for MCF?

What would it look like to use a cut to verify an upper bound on max concurrent flow rate?



All edges have capacity 1.

For an edge set A we say
A separates s_i from t_i
if every $P \in \text{Paths}(s_i, t_i)$ intersects A.

$$\text{sep}(A) = \{ i \mid A \text{ separates } s_i \text{ from } t_i \}.$$

$$\text{cap}(A) = \sum_{(u,v) \in A} c(u,v)$$

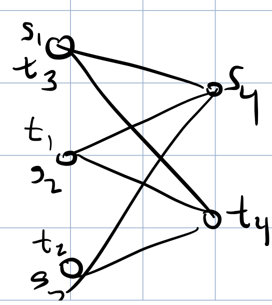
$$\text{Sparsity}(A) = \frac{\text{cap}(A)}{|\text{sep}(A)|}.$$

The sparsity of any cut is an upper bound on the concurrent flow rate of any MCF.

$\therefore$

$$(\text{max concurrent flow rate}) \leq (\text{min sparsity})$$

The two sides aren't always equal.
Counterexample:



Case analysis: Sparsest cut has sparsity 1.
MCF rate $= 3/4$.

(Each flow path uses ≥ 2 edges, so sending at rate r requires

$$\underbrace{2}_{\text{edges per path}} \cdot \underbrace{4}_{\#\text{ of commodities}} \cdot \underbrace{r}_{\text{rate}} \text{ units of capacity.}$$

The graph has only 6 units in total.)

Theorem. (Leighton - Rao) In any undirected graph G, for any set of k source-sink pairs,

$$O(\log k) \cdot \begin{pmatrix} \text{max concurrent} \\ \text{flow rate} \end{pmatrix} \geq (\text{min cut sparsity}).$$

Proof. Plan is to formulate MCF as a linear program, take the dual, round an optimal dual solution to get a sparse cut.

$$\max \quad \rho$$

$$\text{s.t.} \quad \sum_{P \in Paths(s_i, t_i)} f_P \geq \rho \qquad \forall i \in [k]$$

$$\sum_{P : (u,v) \in P} f_P \leq c(u,v) \qquad \forall (u,v) \in E$$

$$f_P \geq 0 \qquad \forall P.$$

$$\max \quad \rho$$

$$\text{s.t.} \quad (d_i) \quad \rho - \sum_{P \in Paths(s_i, t_i)} f_P \leq 0 \qquad \forall i \in [k]$$

$$(\ell_{uv}) \quad \sum_{P : (u,v) \in P} f_P \leq c(u,v) \qquad \forall (u,v) \in E$$

$$f_P \geq 0 \qquad \forall P.$$

$$\min \sum_{(u,v) \in E} c(u,v) \cdot \ell_{uv}$$

$$\text{s.t.} \quad \sum_{(u,v) \in P} \ell_{uv} - d_i \geq 0 \qquad \forall P \in Paths(s_i, t_i)$$

$$\sum_{i=1}^{k} d_i \geq 1$$

$$\ell_{uv}, d_i \geq 0 \qquad \forall (u,v) \in E, \ \forall i \in [k]$$

Interpretation:

ℓ_{uv} is "length" of edge (u,v)

d_i is "distance" from s_i to t_i, along a shortest path.

If A is any edge set and $j = |\text{sep}(A)|$,

then set $\ell_{uv} = \frac{1}{j}$ $\forall$ $(u,v) \in A$.

and $d_i = \frac{1}{j}$ $\forall$ $i \in \text{sep}(A)$.

and all other dual vars zero.

The value of this feasible dual is sparsity (A).

Friday we'll see how to postprocess <u>any</u> dual feasible solution to find a relatively sparse cut.