29 Oct 2021 — Chernoff Bound; Randomized Rounding

Routing to minimize congestion.

Given $G = (V, E)$ [directed]

src-destination pairs $\{(s_i, t_i)\}_{i=1}^{k}$ $s_i, t_i \in V$.

edge capacities $c_{uv} \in \mathbb{N}$. $\forall (u,v) \in E$.

<u>Goal:</u> Choose a routing path P_i connecting s_i to t_i for all i.

Define $\text{load}(u,v) = \#\{ i \mid P_i \text{ contains } (u,v)\}$.

minimize $\max_{(u,v)\in E} \left\{ \frac{\text{load}(u,v)}{c_{uv}} \right\}$.

This is NP-Hard.
The following relaxation is solvable in poly time:

Choose a prob distrib. f_i over paths joining s_i to t_i

Define $\quad \text{load}(u,v) = \sum_{i=1}^{k} \Pr\left((u,v) \text{ belongs to } P_i \text{ sampled from } f_i\right)$

minimize $\quad \max_{(u,v)\in E} \left\{ \frac{\text{load}(u,v)}{c_{uv}} \right\}$.

<u>Why easy?</u> f_i (prob distrib over paths) can be summarized by

$$\phi_i(u,v) = \Pr\left((u,v) \in P_i \text{ sampled from } f_i\right) - \Pr\left((v,u) \in P_i \text{ samp from } f_i\right).$$

Then ϕ_i is a flow of value <u>1</u> from s_i to t_i.

This reduces to solving a poly-size LP.
Variables are $\{\phi_i(u,v)\}\ \forall i,u,v$; r.

minimize $\quad r$

s.t. $\quad \phi_i(u,v) = -\phi_i(v,u) \quad \forall i,u,v.$

$$\sum_v \phi_i(u,v) = \begin{cases} 1 & \text{if } u = s_i \\ -1 & \text{if } u = t_i \\ \emptyset & \text{if } u \notin \{s_i, t_i\} \end{cases} \quad \forall i, u$$

$$\sum_i \phi_i(u,v) \le r \cdot c_{uv} \quad \forall (u,v) \in E.$$

Solve this LP. Obtain flows $\phi_1, \ldots, \phi_k$.
Decompose each ϕ_i as prob distrib over s_i-t_i paths, f_i.

Draw one random sample, P_i, from each distribution f_i, independently.

indep. randomized rounding

Output $(P_1, \ldots, P_k)$.

For any (u,v),

$$\mathbb{E}[\text{load}(u,v) \text{ from } P_1, \ldots, P_k]$$
$$= \sum_{i=1}^{k} \mathbb{E}[\text{load } (u,v) \text{ from } P_i]$$
$$= \sum_{i=1}^{k} \Pr((u,v) \in P_i)$$
$$= \sum_i \phi_i(u,v)$$

(assuming w.l.o.g. $\Pr(\cancel{(v,u)} \in P_i) = 0$)

$$\leq r \cdot c_{uv}$$

Chernoff Bound: If $X_1, \ldots, X_N$ are independent $[0,1]$-valued rand vars, and if $X = X_1 + \ldots + X_N$

$$x = E[X].$$

then $\Pr(X > (1+\varepsilon)x)$ and

$$\Pr(X < (1-\varepsilon)x)$$

are both $< \exp(-\psi(\varepsilon) \cdot x)$.

where $\psi(\varepsilon) > 0$ for $\varepsilon > 0$,

$\psi(\varepsilon) > \frac{1}{3}\varepsilon^2$ for $\varepsilon < 1$.

"The tail probability, i.e. $\Pr(X > (1+\varepsilon)x)$ or $\Pr(X < (1-\varepsilon)x)$ is exponentially small when the expectation of the sum, x, is much bigger than the max possible value of any constituent of the sum."